Fractal Geometry

Special Topics in Dynamical Systems — WS2020

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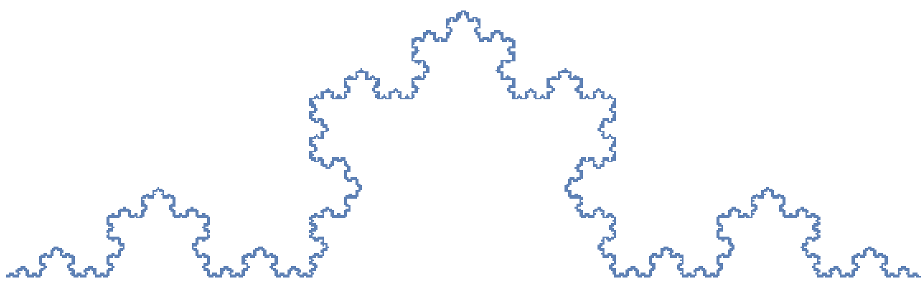
Abstract

Classical shapes in geometry – such as lines, spheres, and rectangles – are only rarely found in nature. More common are shapes that share some sort of "self-similarity". For example, a mountain is not a pyramid, but rather a collection of "mountain-shaped" rocks of various sizes down to the size of a grain of sand. Without any sort of scale reference, it is difficult to distinguish a mountain from a ragged hill, a boulder, or ever a small uneven pebble. These shapes are ubiquitous in the natural world from clouds to lightning strikes or even trees. What is a tree but a collection of "tree-shaped" branches?

A central component of fractal geometry is the description of how various properties of geometric objects scale with size. Dimension theory in particular studies scalings by means of various dimensions, each capturing a different characteristic. The most frequent scaling encountered in geometry is polynomial scaling (e.g. surface area and volume of cubes and spheres) but even natural measures can simultaneously exhibit very different behaviour on an average scale, fine scale, and coarse scale. Dimensions are used to classify these objects and distinguish them when traditional means, such as cardinality, area, and volume, are no longer appropriate. We shall establish fundamental results in dimension theory which in turn influence research in diverse subject areas such as combinatorics, group theory, number theory, coding theory, data processing, and financial mathematics. Some connections of which we shall explore. [1]



[1] I have no doubt that there are many typos and inaccuracies in this manuscript. If you find anything that needs correcting, please let me know at maths@troscheit.eu. Thank you!

Contents

Figure 1: The construction of the Cantor middle-third set.

1 Introduction

Fractal geometry is a relatively young field of mathematics that studies geometric properties of sets, measures, and other structures by identifying recurring patterns at different scales. These objects appear in a great host of settings and fractal geometry links with many other fields such as geometric group theory, geometric measure theory, metric number theory, probability, amongst others. Invariably linked with fractal geometry is dimension theory, which studies the scaling exponents of properties.

In this course we will investigate sets and measures, usually living in $\mathbb{R}^d$, that have these repeating patterns and provide applications to other fields. While we will predominately work in $\mathbb{R}^d$, many results easily extend to much more general metric spaces. In several places, especially in the beginning we give an indication of how far it can be generalised. Another reason to restrict oneself to Euclidean space is visualisation. Fractal geometry is particularly suited for providing "proof by pictures" as a shortcut to understanding geometric relations. For example, it is easy to see that the shapes in Figures 2, 3, 4, and 5 (the Sierpiński gasket, Cantor four corner dust, von Koch curve, and Menger sponge, respectively) are composed of a finite number of similar copies of itself. Many of these objects can be constructed by successively deleting subsets. The Cantor middle-third set is constructed from the unit line by successively removing the middle-third of remaining construction intervals, see Figure 1. The Menger sponge and Sierpiński carpet can be constructed in a very similar manner.

Some of the fundamental questions investigated by fractal geometry are:

- How can we describe and formalise "self-similarity"?
- How "big" are irregular sets?
- How "smooth" are singular measures?

A naïve approach to determining the size of sets, and one that works well in (axiomatic) set theory is their cardinality. Clearly,

$$\{A \subseteq \mathbb{R}^d : A \text{ finite}\} \subset \{A \subseteq \mathbb{R}^d : A \text{ countable}\} \subset \{A \subseteq \mathbb{R}^d\} =: \mathcal{P}(\mathbb{R}^d)$$

but this begs the question of how we differentiate within classes. For finite sets cardinality works well enough, whereas we will need a much more geometric approach to differentiate between sets such as $\mathbb{Q}$ and $\{1/n : n \in \mathbb{N}\}$. One such way is to take local densities to account.

For subsets of $\mathbb{R}^d$, the d-dimensional Lebesgue measure provides a first approach. This limits one to measurable sets (say Borel sets), which we are fine with. However, this classification makes many Lebesgue null sets equivalent. For our purposes, the Lebesgue measure is not fine enough as it does not provide a good "measuring stick" for highly irregular sets such as the Sierpiński gasket (Figure 2) and the Cantor middle-third set (Figure 1). The

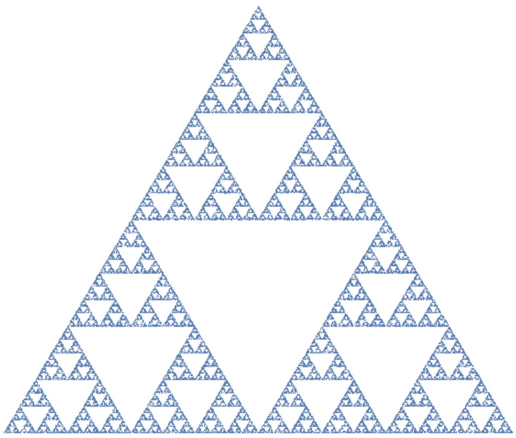


Figure 2: Sierpiński gasket (or triangle). A set exhibiting self-similarity.

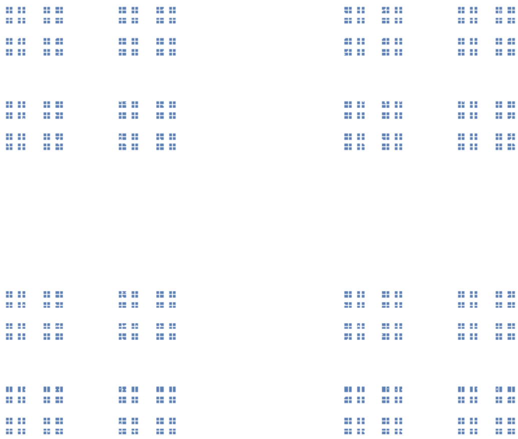


Figure 3: Cantor four corner dust.

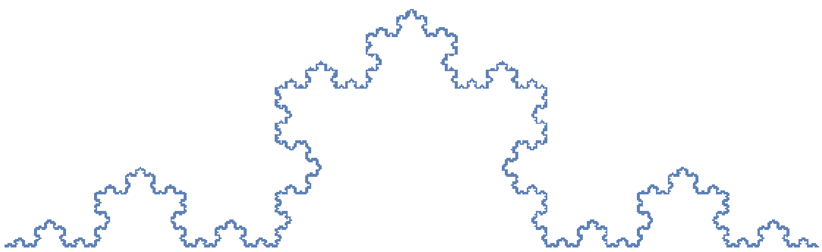


Figure 4: The von Koch curve.

one dimensional Lebesgue measure $\mathcal{L}$ of the Cantor set C is bounded above by the Lebesgue measure of each construction step. Hence,

$$\mathcal{L}(C) \leq 1 - \frac{1}{3} - \frac{2}{3^2} - \cdots - \frac{1}{3}\left(\frac{2}{3}\right)^k = 1 - \frac{1}{3}\frac{(2/3)^{k+1} - 1}{-1/3} = \frac{2^{k+1}}{3^{k+2}}$$

for all $k \in \mathbb{N}$ and so $\mathcal{L}(C) = 0$. For the Sierpiński gasket S we can consider the circumferences of the equilateral triangles in the construction and find that $\mathcal{L}^1(S) \geq 3 + 3/2 + \cdots + (3/2)^k \cdots = \infty$. However, the two-dimensional Lebesgue measure of the Sierpiński gasket can be bounded with a little work by

$$\mathcal{L}^2(S) \leq A\left(1 - \tfrac{1}{4} - \tfrac{3}{16} - \cdots - \tfrac{3^{k-1}}{4^k}\right)$$

where A is the area of an equilateral triangle of sidelength 1. Since this bound holds for all k, we establish that $\mathcal{L}^2(S) = 0$.

We will later generalise the Lebesgue measure to the s-dimensional Hausdorff measure, which is the "correct" measure to look at in many geometric settings. It has established itself as the "gold standard" and we will investigate it in much depth. One of its nice properties is translation invariance as well as scaling appropriately: let S be a similarity, i.e. a map such that there exists $c > 0$ with $|S(x) - S(y)| = c|x - y|$ for all $x, y \in \mathbb{R}^d$. Then, $\mathcal{H}^s(S(E)) = c^s\,\mathcal{H}^s(E)$. This property is especially useful if the measure is positive and finite and one of our goals in this course it to give sufficient conditions when there exists s such that the measure is positive and finite. If there exists such an s, it must be unique. The Hausdorff measure has the property that there exists a unique value s with the property that $\mathcal{H}^t(E) = 0$ for $t > s$ and $\mathcal{H}^t(E) = \infty$ for $0 < t < s$ (assuming $s \neq 0$). This unique value is called the Hausdorff dimension of the set E and the Hausdorff measure at this critical value may take any value in $[0, \infty]$. Much research is devoted to finding not just the dimension, but also bounds on the actual value of the measure for interesting sets.

We can use this scaling property to determine the Hausdorff dimension of sets such as the Sierpiński gasket. Under the assumption[2] that there is an exponent for which the Hausdorff measure is positive and that the s-Hausdorff measure of a single point is zero[3] for $s > 0$, we use the scaling in the following way. Note that the Sierpiński gasket S is constructed of three copies of S scaled by $1/2$ and translated appropriately. Since $\mathcal{H}^s$ is a measure, we get $\mathcal{H}^s(S) = 3\,\mathcal{H}^s(1/2 \cdot S) = 3c^s\,\mathcal{H}^s(S)$. Dividing by the measure (which we assume to be positive and finite) and solving for s gives $s = \log 3/\log 2 = \log_2 3$. Indeed, the Hausdorff measure $\mathcal{H}^{\log_2 3}(S)$ is positive and finite and the above method is justified. We will later see a general method for establishing this.

Other nice properties include the d-dimensional Hausdorff measure being comparable to the d-dimensional Lebesgue measure.

The other dimensions we will consider are the box-counting dimension, the packing dimension, multifractal spectra, and the Assouad dimension. All with their own topological information. Often these dimensions coincide and determining when they do (or do not) gives detailed information about homogeneity and regularity.

Acknowledgements

I am grateful to Max Auer and Silvia Radinger for the correction of many inaccuracies in earlier versions of these lecture notes.

[2] This is quite a big assumption that we have not justified here in any way.

[3] This follows easily from the definition we will see later.

2 Classes of "fractal" sets and measures

We start by introducing some of the most important classes of "fractal" sets. In later chapters we will discuss their properties in full. For now, we will state their definitions, show that they are well-defined, and give a basic overview of their relations to each other.

2.1 middle-α Cantor sets

Similarly to the Cantor middle-third set, we can define a class of subsets of $[0,1]$ by "cutting out" the middle-α of each construction interval.

Let $0 < \alpha < 1$. Let $\{0,1\}^{\mathbb{N}}$ be the collection of binary codings[4]. That is, all infinite strings consisting of the letters 0 and 1. Similarly, $\{0,1\}^k$ are finite strings of length k and we denote the empty word (a word of length 0) by $\emptyset$. Note that the collection of all finite strings defines a semigroup under composition, with $\emptyset$ being the identity. We set $I_\emptyset = [0,1]$ to be the initial construction interval and inductively create the construction intervals I_v, $v \in \{0,1\}^k$ by setting I_{v0}, I_{v1} to be the closed subintervals of I_v by removing the open middle interval of length $\alpha\,\mathcal{L}(I_v)$ of I_v. It is easy to check that $\mathcal{L}(I_v) = ((1-\alpha)/2)^k$, where $v \in \{0,1\}^k$.

These construction intervals are combined to give the level sets

$$C_\alpha^k = \bigcup_{v\in\{0,1\}^k} I_v.$$

The middle-α Cantor set is then given by their intersection

$$C_\alpha = \bigcap_{k\in\mathbb{N}} C_\alpha^k.$$

We note that all construction intervals are compact sets. Therefore the finite unions C_α^k are compact, and indeed C_α is compact. In fact, this class of sets has several nice properties:

1. C_α is closed and therefore compact,
2. C_α is perfect, i.e. it is closed and has no isolated points,
3. C_α is nowhere dense, i.e. its closure has empty interior,
4. C_α is uncountable as there is an injection from the set of binary codes to C_α given by $\Pi : \{0,1\}^{\mathbb{N}} \to C_\alpha$ with $v \mapsto \bigcap_{k\in\mathbb{N}} I_{v|_k}$, where $v|_k$ is the restriction to the initial k letters.
5. The map Π is in fact a bijection,
6. C_α is Lebesgue-null ($\mathcal{L}(C_\alpha) = 0$).
7. C_α is made of two translated and rescaled copies of itself, where the rescaling factor is $(1-\alpha)/2$.

Properties (2) and (3) together are our definition of a Cantor set.

Definition 2.1. *Let $(C, \mathcal{O})$ be a topological space. We say that C is a **(topological) Cantor set** if C is perfect and nowhere dense.*

[4]Identifying point in a set with some abstract coding space will become the norm later on.

Exercise 2.1. *Prove all of the previously mentioned properties of the middle-α Cantor set.*

Using the method established in the introduction we can make a guess to the Hausdorff dimension.

Proposition 2.2. *Let $0 < \alpha < 1$ and C_α be the middle-α Cantor set. Assume that $\mathcal{H}^s(C_\alpha)$ is positive and finite at its Hausdorff dimension $s = \dim_H C_\alpha$. Then,*

$$0 < \dim_H C_\alpha = \frac{\log 2}{\log 2 - \log(1-\alpha)} < 1.$$

Proof. The set C_α consists of two similar copies of itself both scaled by a factor of $(1-\alpha)/2$. In fact, it can be checked that

$$C_\alpha = \left(\tfrac{1-\alpha}{2} \cdot C_\alpha\right) \cup \left(\tfrac{1-\alpha}{2} \cdot C_\alpha + 1 - \tfrac{1-\alpha}{2}\right).$$

Since the Hausdorff measure is translation invariant and scales with exponent s, we have

$$\mathcal{H}^s(C_\alpha) = 2\,\mathcal{H}^s\left(\tfrac{1-\alpha}{2} \cdot C_\alpha\right) = 2\left(\tfrac{1-\alpha}{2}\right)^s \mathcal{H}^s(C_\alpha)$$

and using the assumption that the Hausdorff measure for C_α is positive and finite,

$$1 = 2\left(\tfrac{1-\alpha}{2}\right)^s \quad \Rightarrow \quad \log(1/2) = s \log \tfrac{1-\alpha}{2} \quad \Rightarrow \quad s = \frac{\log 2}{\log 2 - \log(1-\alpha)}$$

as required. □

From the formula we can immediately deduce that $s(\alpha) : (0,1) \to (0,1), \alpha \mapsto \dim_H C_\alpha$ is continuous and $s(\alpha) \to 0$ as $\alpha \to 1$ and $s(\alpha) \to 1$ as $\alpha \to 0$.

Now consider the question of convergence of the sets themselves. On the one hand, the only points that all C_α share are the endpoints $\{0,1\}$ and it is not too unreasonable to think that this convergence should give $C_\alpha \to \{0,1\}$ as $\alpha \to 1$. Similarly, we are taking less and less away from the intervals when $\alpha \to 0$ and we might say $C_\alpha \to [0,1]$ as $\alpha \to 0$. This is in fact the convergence we will formalise, though it does not come without issues. We can clearly see that cardinality is not preserved under this convergence: C_α is uncountable but $\{0,1\}$ is finite. Even being a Cantor set is not preserved: $\{0,1\}$ has isolated points and is not perfect, whereas $[0,1]$ has non-empty interior and so not nowhere dense.

Before defining this convergence we briefly talk about a generalisation of the construction above to include all compact metric spaces.

2.2 Moran sets

A Moran set generalises the construction we saw for the middle-α Cantor set. It is so general, in fact, that any compact metric space is (at least trivially) a Moran set.

Definition 2.3. *Let $M_{0,0}$ be a nonempty compact metric space. For all $n \in \mathbb{N}$, let I_n be a finite index set. Let $\{M_{n,i}\}_{n\in\mathbb{N}, i\in I_n}$ be a collection of nonempty compact metric spaces such that*

$$\forall n \in \mathbb{N},\ \forall i \in I_n,\ \exists j \in I_{n-1} \quad (M_{n,i} \subseteq M_{n-1,j}).$$

*The **Moran set** associated with construction $\{M_{n,i}\}_{n\in\mathbb{N}, i\in I_n}$ is the nonempty compact set*

$$M = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in I_n} M_{n,i}.$$

Immediately we see that any compact metric space can be realised by such a construction, letting $I_n = \{0\}$ and $M_{n,0} = X$ for all n. The fact that M is compact follows from the fact that $M_{n,i}$ are compact and finite unions and arbitrary intersections of compact sets are compact. Nonemptiness follows from Cantor's intersection theorem and the fact that closedness follows from compactness in metric spaces[5].

Theorem 2.4 (Cantor's intersection theorem)**.** *Let $(X, \mathcal{O})$ be a topological space. Let $X_i \subseteq X$ be a sequence of non-empty compact, closed subsets satisfying*

$$X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq \ldots .$$

Then

$$\bigcap_{n \in \mathbb{N}} X_n \neq \varnothing.$$

Proof. Assume for a contradiction that $\bigcap X_n = \varnothing$. Then $U_n = X_1 \setminus X_n$ is open as X_n is closed in X and therefore also as a subspace of X_1. Further

$$\bigcup U_n = \bigcup (X_1 \setminus X_n) = X_1 \setminus \Big(\bigcap X_n \Big) = X_1 \tag{2.1}$$

is an open cover of X_1. By compactness there exists a finite subcover $\{U_{n_1}, \ldots, U_{n_k}\}$ and by nesting, $U_{n_k} \supseteq U_{n_i}$ for all $1 \leq i \leq k$. Therefore, using (2.1), $U_{n_k} = X_1$ and $X_{n_k} = X_1 \setminus U_{n_k} = \varnothing$, a contradiction. $\square$

The trick in using this construction is setting it up in the right way so that we know scaling information between successive levels of the construction. This can be used to find dimension information. Often these sort of sets are used to construct examples of sets with pre-prescribed information and we will use it to construct several counterexamples.

2.3 Invariant sets

Probably the most important class of sets (and measures) we will be studying are in the family of invariant sets. These are sets (or measures) whose repeated nature can be explicitly described by a set of functions under which it is invariant.

Let $U \subset \mathbb{R}^d$ be a non-empty open set. Let $\{f_i\}$ be a finite collection of strict contractions on $\overline{U}$, the closure of U. That is, for all $f_i : \overline{U} \to \overline{U}$ there exists $c_i < 1$ such that

$$|f_i(x) - f_i(y)| \leq c_i |x - y| \qquad \text{for all } x, y \in \overline{U}\,.$$

For technical reasons we want to avoid the maps fixing points in the boundary ∂U and further assume that $f_i(\overline{U}) \subseteq U$.

Perhaps surprisingly, there is a unique compact set that is invariant under these maps, as captured by the following fundamental result.

Theorem 2.5 (Hutchinson)**.** *Let $\mathbb{I} = \{f_i\}$ be a finite collection of strict contractions as above. There exists a unique non-empty compact set $E \subset \mathbb{R}^d$ such that*

$$E = \bigcup_i f_i(E),$$

called the ***invariant set*** *of $\mathbb{I}$.*

[5] Later reference to appendix

We are not quite ready to prove this theorem yet as we will need to introduce a metric on compact spaces. However, we can enjoy a few images and examples of invariant sets before we do so.

Example 2.6. An important family of invariant sets are those where the maps are restricted to be similarities on $\mathbb{R}^d$, i.e. those maps f_i for which $|f_i(x)-f_i(y)| = c_i|x-y|$ for all $x, y \in \mathbb{R}^d$. These invariant sets are called **self-similar sets**. In fact, all previous examples (Sierpiński gasket and middle-α Cantor sets) are self-similar sets. A self-similar map can alternatively be written as $f_i(x) = c_i\mathbf{O}_i x + t_i$, where $0 < c_i < 1$ is a scalar, $\mathbf{O}_i$ is an orthogonal transformation (rotations, reflections, etc.), and t_i is a translation. Some more intricate examples are shown in Figures 6 and 7.

Remark 2.7. *Note that invariant sets include sets that we would not commonly call fractals! For example, the unit line* $[0, 1]$ *is invariant under the maps*

$$f_1(x) = x/2 \text{ and } f_2(x) = x/2 + 1/2.$$

Similarly, Hutchinson's theorem only tells us that there is a unique compact set satisfying the invariance! In the example just now, the sets $[0, 1)$*,* $(0, 1]$ *and* $\mathbb{R}$ *are also invariant under* $\{f_1, f_2\}$*.*

Example 2.8. A more general class is obtained by relaxing the restriction of maps from similarities to affine contractions $f_i : \mathbb{R}^d \to \mathbb{R}^d$, i.e. those of the form $f_i(x) = \mathbf{A}_i x + t_i$, where $\mathbf{A}_i \in \mathbb{R}^{d\times d}$ are invertible matrices with (operator) norm $\|A\| < 1$ and t_i are translations. These invariant sets are referred to as **self-affine sets**. This class coincides with self-similar sets in dimension 1, and one commonly makes the assumption that one of the mappings is strictly affine (and thus we consider affine maps in $\mathbb{R}^d$ for $d \geq 2$).

These sets are much more difficult to handle and are a very active area of research with some significant progress made over the last five years. Famous examples are the Bedford-McMullen carpets and higher dimensional analogues. Figure 8 is a Bedford-McMullen carpet where the affine contraction is given by

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}$$

with translations $t_1 = (0, 0)$, $t_2 = (1/2, 0)$, $t_3 = (0, 2/3)$, and $t_4 = (1/2, 1/3)$. The Barnsley fern, Figure 9, is another example of a self-affine set invariant under four maps.

Example 2.9. Both self-similar and self-affine sets are defined by linear maps and thus are very rigid in construction. Much of this rigidity is not needed and many of the results that hold for self-similar sets also hold for the class of self-conformal sets. The class of **self-conformal sets** are sets invariant under conformal (i.e. angle preserving) maps in $\mathbb{R}^d$ for $d \geq 2$, often interpreted as conformal maps on $\mathbb{C}$. A simple example is the upper semi-circle $\mathcal{C} \subset \mathbb{C}$, which is invariant under the maps $f_1(z) = \sqrt{z}$ and $f_2(z) = i\sqrt{z}$, where $\sqrt{\cdot}$ is $z = re^{i\pi\theta} \mapsto \sqrt{r} \cdot e^{i\pi\theta/2}$. Because of the singularity of the derivative at 0, we need to restrict the domain to $\mathbb{C} \setminus B(0, \varepsilon)$, where $0 < \varepsilon < 1$.

Other examples include some Julia sets of the dynamical system $z \mapsto z^2 + c$ (Figure 11) as well as invariant sets under Möbius transformations (Figure 10).

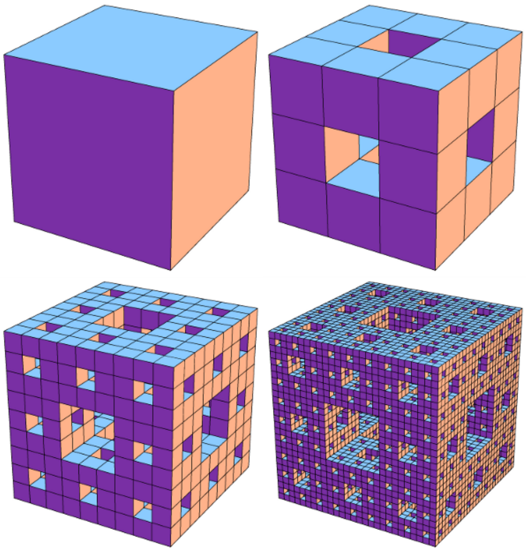


Figure 5: The Menger Sponge.

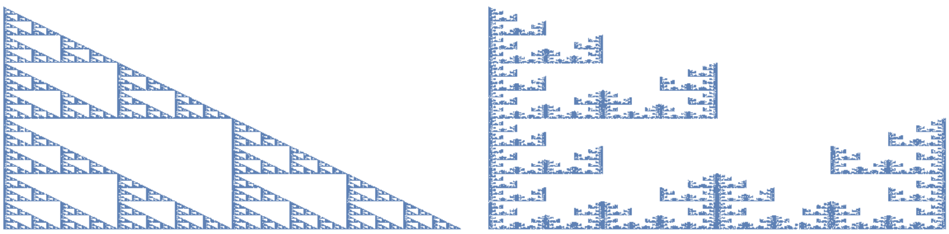


Figure 6: Two variants of the Sierpiński gasket.

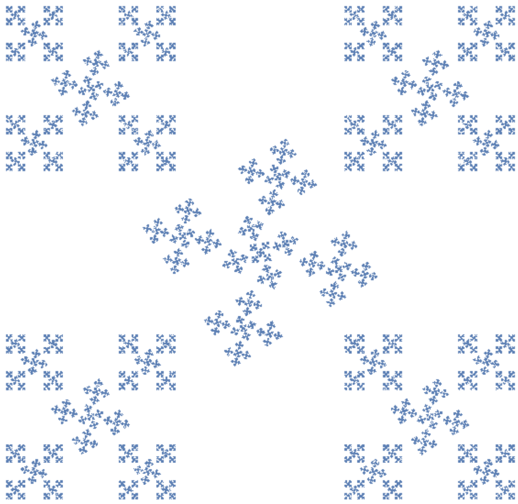


Figure 7: A variant of Cantor dust with central rotation.

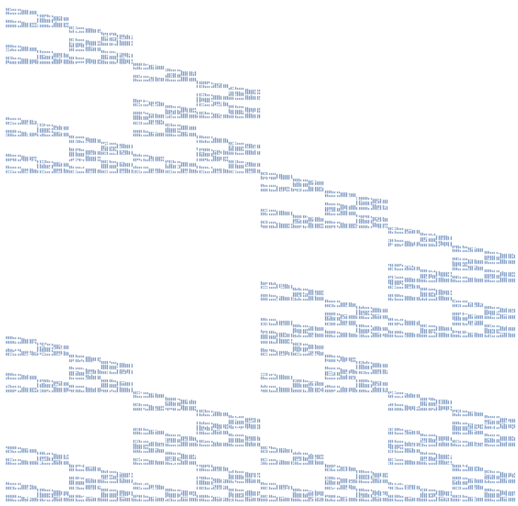


Figure 8: The Bedford McMullen carpet.

Figure 9: The Barnsley fern.

Figure 10: A self-conformal set invariant under three Möbius transformations.

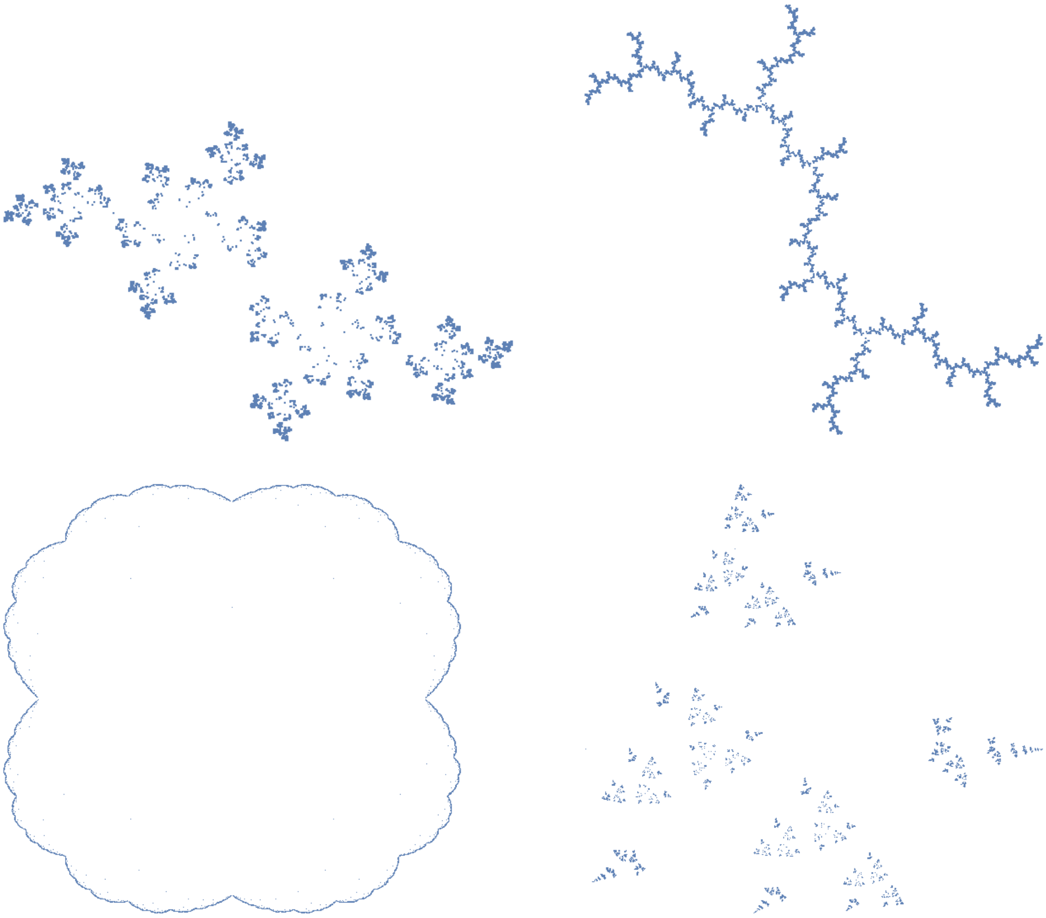


Figure 11: Complex sets.

2.3.1 A metric on the set of compact sets

We previously hinted at developing a useful metric between subsets of $\mathbb{R}^d$. This natural metric is given by the Hausdorff distance d_H which is a useful metric between subsets of the same space.

Definition 2.10. *Let (X, d) be a metric space. The **Hausdorff distance** of two subsets $A, B \subseteq X$ is*

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \ \sup_{b\in B} \inf_{a\in A} d(a, b)\}.$$

So, given two sets A, B that are at Hausdorff distance δ, the definition says that for any point $a \in A$ there exists a point $b \in B$ that is less than $\delta + \varepsilon$ away, where $\varepsilon > 0$ is arbitrary. Further, if the sets are complete, we can take $\varepsilon = 0$. The relation is symmetric and therefore the same will hold for any point $b \in B$ there is a point in A that is $\delta + \varepsilon$ close. The distance between two sets is therefore the minimal distance for which every point in one set has a corresponding point in the other.

The Hausdorff distance of two arbitrary non-empty subsets $A, B \subseteq X$ is well-defined (though may be infinite). However, it is generally not a metric on $\mathcal{P}(X) = \{A \subseteq X\}$, the power set of X. For instance, letting $X = \mathbb{R}$ the sets $A = [0,1]$ and $B = (0,1)$ satisfy $d_H(A, B) = 0$ but are not equal. It turns out that d_H is however a pseudo-metric on $\mathcal{P}(X) \setminus \{\varnothing\}$.

Lemma 2.11. *Let (X, d) be a metric space. The Hausdorff distance d_H is a pseudo metric on $\mathcal{P} \setminus \{\varnothing\}$.*

Proof. That $d_H(A, A) = 0$ and $d_H(A, B) = d_H(B, A)$ follow directly from the definition. It remains to prove the triangle inequality. Let $A, B, C \subseteq X$ be non-empty. Let $a \in A, b \in B, c \in C$. Then $d(a, c) \leq d(a, b) + d(b, c)$ as d is a metric on X. Therefore,

$$\sup_{a\in A} \inf_{c\in C} d(a,c) \leq \sup_{a\in A} \inf_{c\in C} (d(a,b) + d(b,c))$$

$$= \sup_{a\in A} d(a,b) + \inf_{c\in C} d(b,c) \leq \sup_{a\in A} d(a,b) + \sup_{b^*\in B} \inf_{c\in C} d(b^*,c)$$

for all $b \in B$. Minimising $d(a, b)$, we get

$$\sup_{a\in A} \inf_{c\in C} d(a,c) \leq \sup_{a\in A} \inf_{b\in B} d(a,b) + \sup_{b^*\in B} \inf_{c\in C} d(b^*,c)$$

and analogously,

$$\sup_{c\in C} \inf_{a\in A} d(a,c) \leq \sup_{c\in C} \inf_{b\in B} d(a,b) + \sup_{b^*\in B} \inf_{a\in A} d(b^*,c).$$

Hence,

$$d_H(A,C) \leq d_H(A,B) + d_H(B,C)$$

as required. □

Since d_H is a pseudo-metric we can force it to be a metric on $\mathcal{P}$ by quotienting out the sets of distance 0. However there is a simpler way of getting a metric: we assume that (X, d) is complete and define the metric on all complete subsets of X.

Exercise 2.2. *Let (X, d) be a complete metric space. Show that d_H is a metric on the set of all complete subsets of X.*

As mentioned before, we usually are content with looking at subsets of $\mathbb{R}^d$ as Euclidean space has nice properties. However, for the remainder we really only need the space to be complete and locally totally bounded. This is because the generalisation of the Heine-Borel theorem holds in these sets and compact is equivalent to a subset being complete and totally bounded.

We establish a technical lemma that show that the space of all compact subsets of a complete locally totally bounded metric space is also a nice space.

Lemma 2.12. *Let (X, d) be a complete metric space that is locally totally bounded.*[6] *Then the space of compact subsets $\mathcal{K}(X)$ endowed with the Hausdorff metric is complete.*[7]

[6] A space X is **locally totally bounded** if every ball $B = B(x, R) \cap X$ can be covered by finitely many balls of radius r, for all $r > 0$.

[7] This lemma remains true, even if we remove the locally totally bounded assumption. However, as it is much fiddlier we only prove this restriction, which is more than enough for our purposes.

Proof. We first remark that any bounded subset of X is totally bounded by virtue of X being locally totally bounded. The following are equivalent, due to the Heine-Borel theorem:

- $K \subseteq X$ is compact.
- $K \subseteq X$ is closed and bounded.
- K is sequentially compact (every sequence in K has a convergent subsequence).

Further, a subset $A \subseteq X$ is closed if and only if it is complete.

Let $K_i \in \mathcal{K}(X)$ and assume (K_i) is a Cauchy sequence with respect to d_H. We need to show that there exists $K \in \mathcal{K}(X)$ such that $d_H(K_i, K) \to 0$ as $i \to \infty$. We shall verify that this set is

$$K = \{x \in X : \exists x_i \in K_i \text{ such that } x_i \to x\}.$$

First we show that K is complete. Let $k_n \in K$ be a Cauchy sequence. Since $k_n \in K$ there exists $x_{n,i} \in K_i$ such that $x_{n,i} \to k_n$ as $i \to \infty$. Let $I_n \in \mathbb{N}$ be large enough so that $d_H(K_i, K_j) \leq 1/n$ for all $i, j \geq I_n$, and $d(x_{n,I_n}, k_n) \leq 1/n$, as well as $I_n > I_{n-1}$. It can be checked that I_n can be chosen in such a way and that it partitions $\mathbb{N}$:

$$\mathbb{N} = \{\underbrace{1, 2, \ldots, I_1 - 1}_{\text{first partition}}, \underbrace{I_1, I_1 + 1, \ldots, I_2 - 1}_{\text{second partition}}, \underbrace{I_2, \ldots}_{\text{etc.}}, I_k, \ldots\}.$$

For $m \in [1, I_1)$ choose $y_m \in K_m$ arbitrary. For $m \in [I_n, I_{n+1})$ choose $y_m \in K_m$ such that $d(y_m, x_{n,I_n}) \leq 1/n$ (which we can as $d_H(K_{I_n}, K_m) \leq 1/n$ and K_m is compact). Note that $d(y_m, k_n) \leq 2/n$ by the triangle inequality. As X is complete, $k_n \to k$ for some $k \in X$. But $d(k, y_m) \leq d(k, k_n) + 2/n \to 0$ as $n \to \infty$ and so $y_m \in K_m$ converges to k. By definition of K, we also have $k \in K$. So K is complete and hence closed.

Note that for large enough n, the set K_j $(j \geq I_n)$ is contained in a fattening of K_{I_n},

$$K_j \subseteq \{x \in X : \inf\{d(x, y) : y \in K_{I_n}\} \leq 1\} =: [K_{I_n}]_1.$$

So $K \subseteq \overline{[K_{I_n}]}_1$ and K is bounded since K_{I_n} is bounded. Using the equivalency at the start of the proof we find that K is compact. We still need to check that K is non-empty, see exercise below.

Lastly, we need to verify that $d_H(K_i, K) \to 0$. Assume for a contradiction that it does not converge. Then there exists a sequence n_k such that $d_H(K_{n_k}, K) \geq \delta$ for some $\delta > 0$. There are two cases two consider:

1. There are infinitely many n_k such that there exists $x_{n_k} \in K$ but $\inf_{y \in K_{n_k}} d(x_{n_k}, y) \geq \delta$.
2. There are infinitely many n_k such that there exists $x_{n_k} \in K_{n_k}$ but $\inf_{y \in K} d(x_{n_k}, y) \geq \delta$.

Case 1: Since K is compact there exists a convergent subsequence $x_{n_{k_i}} \to x$ with $B(x, \delta/2) \cap K_{n_{k_i}} = \varnothing$. But then x cannot be an accumulation point of $x_m \in K_m$ and $x \notin K$, a contradiction.

Case 2: Let $I_0 = n_k$ for k large enough such that $d_H(K_i, K_j) \leq \delta/3$ for all $i, j \geq I_0$. For $n \in \mathbb{N}$ pick I_n large enough such that $I_{n+1} > I_n$ and $d_H(K_i, K_j) \leq (1/2)^n \delta/3$. Then pick $x_i \in K_i$ arbitrary for $i < I_0$, pick x_{I_0} such that $\inf_{y \in K} d(x_{I_0}, y) \geq \delta$, and for $I_n < i \leq I_{n+1}$ choose $x_i \in B(x_{I_n}, (1/2)^n \delta/3) \cap K_i$. This is a Cauchy sequence (check!) and thus converges to some $x \in X$. Clearly,

$$d(x, x_{I_0}) \leq \sum_{n=0}^{\infty} d(x_{I_n}, x_{I_{n+1}}) \leq \tfrac{2}{3}\delta.$$

So $\inf_{y\in K} d(x,y) \geq \delta/3$ and $x \in K$ by definition of K. This is however a contradiction and our claim follows. $\square$

We are now going to prove that there exists a single non-empty compact set that is invariant under sets of contractions. Like many existence and uniqueness proofs it is based on Banach's fixed point theorem. The trick is to set up the right space on which to apply the theorem.

Theorem 2.13 (Banach fixed point theorem)**.** *Let (X,d) be a metric space and $T : X \to X$ be a contraction. Then there exists a unique $x_0 \in X$ such that $T(x_0) = x_0$.*

Equipped with this we are ready to prove Theorem 2.5.

Proof of Theorem 2.5. By assumption $\overline{U}$ is closed and therefore complete, further $\mathbb{R}^d$ is locally totally bounded and so is $\overline{U}$. We can apply Lemma 2.12 and conclude that $(\mathcal{K}(\overline{U}), d_H)$ is a complete metric space. We define a map on $\mathcal{K}(\overline{U})$ called the **Hutchinson operator** $H : \mathcal{K}(\overline{U}) \to \mathcal{K}(\overline{U})$,

$$H(K) = \bigcup_i f_i(K).$$

Since all f_i maps $\overline{U}$ into itself this map is well defined. To apply Banach's fixed point theorem we need to show that H is contracting.

Let $A, B \in \mathcal{K}(\overline{U})$. Since both sets are compact they are bounded and $d_H(A,B) < \infty$. If $d_H(A,B) = 0$ then $A = B$ and so $H(A) = H(B)$ and $d_H(H(A), H(B)) = 0$. Therefore H is trivially a contraction and we assume that $A \neq B$. Then $\delta = d_H(A,B) > 0$ and $\forall a \in A \exists b \in B (d(a,b) \leq \delta)$ by compactness of A and B. Similarly, $\forall b \in B \exists a \in A (d(a,b) \leq \delta)$.

Now consider $A^* = H(A)$ and $B^* = H(B)$. For all $a^* \in A^*$ there exists i and $a \in A$ such that $f_i(a) = a^*$. Further, there exists $b \in B$ such that $d(a,b) \leq \delta$ and upon writing $b^* = f_i(b)$ we get

$$d(a^*, b^*) = d(f_i(a), f_i(b)) \leq c_i d(a,b) \leq c_i \delta.$$

Similarly, for all $b^* \in B^*$ there exists $a^* \in A^*$ and j such that $d(a^*, b^*) \leq c_j \delta$. Therefore

$$d_H(A^*, B^*) = d_H(H(A), H(B)) \leq \max\{c_i\}\delta = c_{\max} d_H(A,B)$$

for $c_{\max} = \max_i c_i < 1$ and H is indeed a contraction. Application of Banach's fixed point theorem now gives the existence and uniqueness of $E \in \mathcal{K}(\overline{U})$ such that $H(E) = E$, as was required. $\square$

Remark 2.14. *Note that this proof will work in much more generality. For example, we could take a countable collection of contractions, as long as their contraction was uniformly bounded away from* 1*. Further, this proof also shows that invariant measures are unique such as the family of Bernoulli measures defined by the invariance $\mu(.) = \sum_i p_i \cdot (\mu \circ f_i(.))$. Here the space of compact sets is replaced by the space of compactly supported probability measures on a complete space with the Wasserstein metric. While we are ignoring these technicalities, we will later deal with invariant measures and take their existence for granted.*

2.4 Quasi self-similar sets

So far we have only seen sets that are invariant under sets of maps. And even though there are many contraction mappings out there, this is still too rigid to properly define the heuristic "self-similarity" we are trying to formalise. Especially since we want to allow sets that look roughly similar on different scales but do not have to look exactly the same. One way of getting around this is to generalise the notion of self-similar sets by capturing the essence of looking roughly similar on different scales. This class of sets is known as the class of quasi self-similar sets, which encompasses self-similar, self-conformal, and many other sets.

Definition 2.15. *Let $E \subset \mathbb{R}^d$ be a non-empty compact set. If there exists $c \geq 1$ such that for every closed ball $B(x,r)$ centred in E (i.e. $x \in E$) of radius $0 < r \leq \operatorname{diam} E$ there exists mapping $g : E \to B(x,r) \cap E$ with*

$$c^{-1} r|y-z| \leq |g(y) - g(z)| \leq cr|y-z| \tag{2.2}$$

*for all $y, z \in E$, we call E **quasi self-similar (QSS)**.*

Remark 2.16. *We note that there are several ways of defining quasi self-similarity and we have opted for the most common definition. A looser definition only requires the lower bound in* (2.2)*. Other definitions consider being able to embed small images into the entire set, as opposed to embedding the entire set into small balls.*

Example 2.17. All self-similar sets are quasi-self similar. This can be immediately deduced from self-similar sets being made up of similar images of the entire set. Hence given any point $x \in E$ and scale $r > 0$, we can find an image $f_{i_1} \circ \cdots \circ f_{i_k}(E)$ that is contained in the ball and gives our embedding g.

This is also true for self-conformal sets, as we will see later on.

Example 2.18. A simple example of quasi self-similar sets that are not invariant can be obtained by choosing a self-similar set, and deleting parts recursively such that $E \supset \bigcup_i f_i(E)$. The deletion can be chosen (e.g. randomly) such that there is no finite collection of functions for which E is invariant. Any set satisfying this weaker form of invariance is known as a **sub self-similar set**.

While the class of quasi self-similar sets is much larger than that of self-similar sets, their geometric properties are still very closely connected and many results that are established for self-similar sets generalise naturally to quasi self-similar sets.

2.5 The universe of "fractal sets"

Given all these definitions, it might help to summarise their connections. Figure ?? contains the "Atlas" of fractal-land.

3 Dimension Theory

Dimension theory studies the scaling of sets and measures by taking some quantity, say coverings, that is dependent on a scale and observe its change with varying scale. In many natural settings this relationship is exponential, i.e. the quantity N_r changes like r^{-d}, where d is the dimension. For instance, the volume of a three-dimensional ball of radius r is proportional to r^{-3}. The simplest way of formalising dimension this way is with the box-counting dimension.

3.1 Box-counting dimension

Consider a bounded subset $X \subset \mathbb{R}^d$ in Euclidean space. We let $N_r(X)$ be the minimal number of r-balls needed to cover X. Since $\mathbb{R}^d$ is a totally bounded space, this number will always be finite.

Using the heuristic above, we expect this quantity to be proportional to r^{-d}, where d is the dimension. Solving $N_r(X) \approx r^{-d}$ for d, we obtain

$$d \approx \frac{\log N_r(X)}{-\log r}.$$

The box-counting dimension is the limiting value of this relationship.

Definition 3.1. *Let $X \subset \mathbb{R}^d$ be bounded. The **upper** and **lower box-counting dimension** are, respectively,*

$$\overline{\dim}_B X = \limsup_{r\to 0} \frac{\log N_r(X)}{-\log r} \quad \textit{and} \quad \underline{\dim}_B X = \liminf_{r\to 0} \frac{\log N_r(X)}{-\log r}.$$

If the limits coincide, we talk of the box-counting dimension $\dim_B X = \overline{\dim}_B X = \underline{\dim}_B X$.

The choice of letting N_r be the minimal number of r-balls covering X is somewhat arbitrary and N_r can be substituted by several different notions. One such quantity is related to the concept of mesh cubes.

Definition 3.2. *Let $r > 0$ and $t = (t_1, \dots, t_d) \in \mathbb{R}^d$. The **tiling of $\mathbb{R}^d$ by r-mesh cubes with offset** t is the set*

$$\mathbf{Q}_r = \left\{ [t_1 + k_1 r,\ t_1 + (k_1+1)r] \times \cdots \times [t_d + k_d r,\ t_d + (k_d+1)r] : (k_1, \dots, k_d) \in \mathbb{Z}^d \right\}.$$

Each $Q \in \boldsymbol{Q}_r$ is referred to as an r-mesh cube.

It is easy to see that the set $\mathbf{Q}_r$ covers $\mathbb{R}^d$, and that intersections of distinct cubes is limited to their boundary. It is customary to leave $t = (0, \dots, 0)$, as translation does not change anything with regards to asymptotic properties.

Proposition 3.3. *Let $X \subset \mathbb{R}^d$ be bounded. Then*

$$\overline{\dim}_B X = \limsup_{r\to 0} \frac{\log M_r(X)}{-\log r} \quad \textit{and} \quad \underline{\dim}_B X = \liminf_{r\to 0} \frac{\log M_r(X)}{-\log r},$$

where M_r is any of:

1. *smallest number of sets with diameter less than r that cover X,*
2. *smallest number of closed balls of radius r that cover X,*
3. *smallest number of (axis aligned) d-dimensional cubes of sidelength r that cover X,*
4. *number of r-mesh cubes that intersect X,*
5. *largest number of disjoint balls of radius r with centres in X.*

Proof. We only show that 4 and 3 are equivalent. Let $M_r(X)$ be the smallest number of d-dimensional cubes of sidelength r that cover X and let $M_r'(X)$ be the number of r-mesh cubes that intersect X. Since the r-mesh cubes that intersect X are d-dimensional cubes and form a cover of X, we trivially have $M_r(X) \le M_r'(X)$. To establish a complementary bound, first note that any d-dimensional cube of sidelength r is of the form $[x_1, x_1 + r] \times \cdots \times [x_d, x_d + r]$. In each coordinate this intersects at most two intervals of the form $[kr, (k+1)r]$ for $k \in \mathbb{Z}$. Therefore, each cube intersects at most 2^d mesh cubes and

$$2^{-d} M_r'(X) \le M_r(X) \le M_r'(X).$$

We conclude that the limits must coincide as $-\log c M_r(X)/\log r = -\log M_r(X)/\log r - \log c/\log r$ and $\log c/\log r \to 0$ as $r \to 0$. □

Exercise 3.1. *Prove the rest of the equivalencies in Proposition 3.3.*

Note that the logarithm in the definition leads to the suppression of subexponential effects. This makes it both easier to find the power law that is in place, but also ignores more subtle effects. For any $f(r)$ such that $\log f(r)/\log r \to 0$ we obtain the same box-counting dimension, where $N_r(X) = f(r)r^{-d}$. In particular, we could have $f(r) \to \infty$ (as long as this is subexponential) and the box-counting dimension is not affected. In fact, we have exploited this in the proof of Proposition 3.3 where we have shown that all the different definitions of $M_r(X)$ are within in a constant of each other. We can also reduce work by showing convergence along a suitable subsequence. Let $c \in (0,1)$ and chose $r_k \to 0$ such that $r_{k+1} \geq cr_k$. Then, for $r \in [r_{k+1}, r_k)$,

$$\frac{\log N_r(X)}{-\log r} \leq \frac{\log N_{r_{k+1}}(X)}{-\log r_k} = \frac{\log N_{r_{k+1}}(X)}{-\log r_{k+1} + \log(r_{k+1}/r_k)} \leq \frac{\log N_{r_{k+1}}(X)}{-\log r_{k+1} + \log c}$$

and so $\limsup_{r\to 0} \frac{\log N_r(X)}{-\log r} \leq \limsup_{k\to\infty} \frac{\log N_{r_k}(X)}{-\log r_k}$. The lower bound follows as r_k is a subsequence and the upper box-counting dimension can be calculated by taking a subsequence that does not decrease too fast.

Exercise 3.2. *Show that the lower box-counting dimension can also be calculated by taking such a subsequence.*

Equipped with the definition and the flexibility of choosing the geometric meaning of N_r, we can calculate the box-counting dimension of many sets. First, a natural upper bound.

Proposition 3.4. *Let $X \subset \mathbb{R}^d$ be bounded. Then $\overline{\dim}_B X \leq d$.*

Proof. Since X is a bounded subset of $\mathbb{R}^d$, there exists $r_0 = 2^k$ for some $k \in \mathbb{N}$ such that X is contained in the cube $Q = [-r_0, r_0]^d$. Let $r_n = r_0/2^n = 2^{k-n}$. Then $N_{r_n}(X) \leq 2^{dn}$ as $X \subseteq Q$ and Q can be covered by 2^{dn} cubes of sidelength r_n. Since $r_{n+1} \geq r_n/2$ we get

$$\overline{\dim}_B X \leq \limsup_{n\to\infty} \frac{\log N_{r_n}(Q)}{-\log r_n} = \limsup_{n\to\infty} \frac{\log 2^{dn}}{\log 2^{n-k}} = d$$

as required. □

Example 3.5. Let B be the unit ball in $\mathbb{R}^3$. Its box-counting dimension is 3.

Proof. From Proposition 3.4, we get the required upper bound. Let $Q = [0, 1/2]^3$. Since the longest diagonal is of length $\sqrt{3/2^2} < 1$ we have $Q \subset B$. Letting $r_n = 2^{-n}$ $(n \geq 1)$, there are 2^{3n} distinct points of form $(k_1/2^n, k_2/2^n)$ $(k \in \mathbb{Z})$ contained in Q. Since these distinct dyadic rationals are at least r_n separated, there are 2^{3n} mutually disjoint open balls of radius r_n centred in Q, and hence B. Thus,

$$\underline{\dim}_B B \geq \liminf_{n\to\infty} \frac{\log 2^{3n}}{-\log 2^{-n}} = 3$$

from which our claim follows. □

Exercise 3.3. *Let $L = [a, b] \subset \mathbb{R}$ be a line segment, calculate its box-counting dimension. Does the dimension change when seen as a subset of $\mathbb{R}^d$? Does the box-counting dimension vary under translation? under isometries?*

Example 3.6. Let C be the middle-third Cantor set. The box-counting dimension of C exists and $\dim_B C = \log 2/\log 3$

Proof. Let $3^{-k} < r \leq 3^{-k+1}$. The 2^k level k construction intervals provide a cover of sets of diameter no larger than r. Thus,

$$\overline{\dim}_B C \leq \limsup_{r\to 0} \frac{\log N_r(C)}{-\log r} \leq \limsup_{k\to\infty} \frac{\log 2^k}{-\log 3^{-k+1}} = \limsup_{k\to\infty} \frac{k}{k-1}\frac{\log 2}{\log 3} = \frac{\log 2}{\log 3}.$$

For the lower bound let k be such that $3^{-k-1} \leq r < 3^{-k}$. The left endpoints of the construction intervals are contained in C and are separated by at least 3^{-k}. Hence there exist at least 2^k mutually disjoint balls of radius r. So,

$$\underline{\dim}_B C \geq \liminf_{r\to 0} \frac{\log \#\{B(x_i, r)\}}{-\log r} \geq \liminf_{k\to\infty} \frac{\log 2^k}{-\log 3^{-k-1}} = \frac{\log 2}{\log 3}. \qquad \square$$

Example 3.7. Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$. The box-counting dimension is $\dim_B X = 1/2$.

Proof. Let $r > 0$ be given. We enumerate $x_0 = 0$ and $x_n = 1/n$ for $n \geq 1$. Consider the difference between successive points in X:

$$|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n^2+n}.$$

Let $K \in \mathbb{N}$ be such that $1/((K+1)^2+K+1) < r \leq 1/(K^2+K)$. For $k \geq K$, $|x_{k+1} - x_k| \leq r$ and we can cover $[0, x_K]$ by x_K/r many r-balls. The remainder of the set, $X \setminus [0, x_K] = \{x_1, \ldots, x_{K-1}\}$ can be covered by $K-1$ balls of radius r. Hence,

$$N_r(X) \leq \frac{x_K}{r} + K - 1 \leq \frac{1}{Kr} + K. \tag{3.1}$$

Using the bounds on r we get

$$r^{-1} < K^2 + 3K + 2 \leq 6K^2 \Rightarrow K^{-1} < \sqrt{6r}$$

and

$$K^2 + K \leq r^{-1} \Rightarrow K \leq r^{-1/2}.$$

Using these bounds in (3.1) gives

$$N_r(X) \leq \sqrt{6} r^{-1/2} + r^{-1/2} \leq 4r^{-1/2}$$

and $-\log N_r(X)/\log r \leq 1/2 - \log 4/\log r \to 1/2$ as $r \to 0$. Therefore $\overline{\dim}_B X \leq 1/2$. The lower bound is left as an exercise. $\square$

Exercise 3.4. *Finish the proof for Example 3.7.*

The last example might be somewhat surprising in light of the discussion in the introduction. X is a countable and compact set, yet it has positive box-counting dimension. This immediately implies that the box-counting dimension is not stable under countable unions, meaning that $\dim_B \bigcup_{i\in\mathbb{N}} X_i \neq \sup \dim_B X_i$. This, in fact, is a great drawback with the box-counting dimension and why the Hausdorff dimension is, in practise, better behaved. However, the box-counting dimension does satisfy the following basic properties we might ask of any reasonable definition of a *dimension*.

Theorem 3.8. *The box-counting dimension satisfies the following basic properties:*

1. ***Monotonicity.*** *If $E \subseteq F$, then $\overline{\dim}_B E \leq \overline{\dim}_B F$ and $\underline{\dim}_B E \leq \underline{\dim}_B F$.*

2. ***Range of values.*** *If $E \subseteq \mathbb{R}^d$ is bounded, then*

$$0 \leq \underline{\dim}_B E \leq \overline{\dim}_B E \leq d.$$

Further, for all $s, t \in [0, d]$ with $s \leq t$ there exists a compact $E \subset \mathbb{R}^d$ such that $\overline{\dim}_B E = t$ and $\underline{\dim}_B E = s$.

3. ***Finite stability.*** *The upper box-counting dimension is finitely stable, that is,*

$$\overline{\dim}_B E \cup F = \max\{\overline{\dim}_B E, \overline{\dim}_B F\}$$

4. ***Open sets.*** *If $F \subset \mathbb{R}^d$ is non-empty and open, then $\dim_B F = d$.*

5. ***Finite sets.*** *If $X \subset \mathbb{R}^d$ is finite, then $\dim_B X = 0$.*

Proof. (1.) Monotonicity follows from the observation that any r-cover of F is an r cover of E.

(2.) The lower bound of zero follows directly from the definition and the fact that N_r is non-negative. The upper bound follows from the boundedness of E and that any d-dimensional ball can be covered by Cr^{-d} r-balls. We postpone a proof for possible values until we have introduced the Hausdorff dimension.

(3.) Finite stability is an exercise.

(4.) If F is non-empty and open there exists $r_0 > 0$ such that $B(x, r_0) \subset F$. Therefore $N_r(F) \geq N_r(B(x, r_0)) \geq c r_0^d r^{-d}$ from which the result follows. □

Exercise 3.5. *Show that the upper box-counting dimension is finitely stable.*

Exercise 3.6. *(difficult) Give an example of two sets $E, F \subset \mathbb{R}$ such that*

$$\underline{\dim}_B E \cup F > \max\{\underline{\dim}_B E, \underline{\dim}_B F\}.$$

Lipschitz maps play an important role in fractal geometry, not least since homeomorphisms are too weak to preserve the notion of dimension. Our notions of dimension behave much better with respect to Lipschitz mappings as the following results show.

Proposition 3.9. *Let $F \subset \mathbb{R}^d$ be bounded and $f : F \to \mathbb{R}^d$ be Lipschitz, that is, there exists $c > 0$ such that*

$$|f(x) - f(y)| \leq c|x - y| \quad \text{for all } x, y \in F.$$

Then $\underline{\dim}_B f(F) \leq \underline{\dim}_B F$ and $\overline{\dim}_B f(F) \leq \overline{\dim}_B F$.

Proof. Let $\{U_i\}$ be a cover of F of sets with diameter at most r. Then $\{U_i \cap F\}$ is also a cover of F of diameter at most r and hence $\{f(U_i \cap F)\}$ is a cover of $f(F)$ of sets with diameter at most cr. Therefore we get the upper bound $N_{cr}(f(F)) \leq N_r(F)$ which after taking the appropriate limits gives the desired bound. □

Proposition 3.10. *Let $F \subset \mathbb{R}^d$ be bounded and $f : F \to \mathbb{R}^d$ be bi-Lipschitz, that is, there exists $c > 0$ such that*

$$c^{-1}|x - y| \leq |f(x) - f(y)| \leq c|x - y| \quad \text{for all } x, y \in F.$$

Then $\underline{\dim}_B f(F) = \underline{\dim}_B F$ and $\overline{\dim}_B f(F) = \overline{\dim}_B F$.

Proof. From the lower bound it is immediate that f is injective and thus has inverse f^{-1} on $f(F)$. Let $x, y \in F$ then $u = f(x)$ and $v = f(y)$ satisfy

$$c^{-1}|f^{-1}(u) - f^{-1}(v)| = c^{-1}|x-y| \leq |f(x) - f(y)| = |f \circ f^{-1}(u) - f \circ f^{-1}(v)| = |u - v|$$

and we see that f^{-1} is Lipschitz also. We can now apply Proposition 3.9 to f and f^{-1} from which our result follows. $\square$

The notion of bi-Lipschitz is thus for our purposes the right invariant function. Every isometry, similarity, and affine map is bi-Lipschitz this tells us that the box-counting dimension is a geometric invariant. We can also apply Proposition 3.9 to projections. Note that any orthogonal projection from $\pi : \mathbb{R}^d \to \mathbb{R}^m$, where $m < d$ cannot increase distances. That is,

$$|\pi(x) - \pi(y)| \leq |x - y|$$

for all $x, y \in \mathbb{R}^d$ and hence the dimension cannot increase under projections.

Corollary 3.11. *Let $F \subset \mathbb{R}^d$ be bounded. Let $\pi : \mathbb{R}^d \to \mathbb{R}^m$ for $m < d$ be an orthogonal projection. Then,*

$$\overline{\dim}_B \pi F \leq \min\{\overline{\dim}_B F, m\} \quad \text{and} \quad \underline{\dim}_B \pi F \leq \min\{\underline{\dim}_B F, m\}.$$

Exercise 3.7. *Let $g(x)$ be a differentiable function on $[0,1]$ with $\sup_{x\in[0,1]} g'(x) < \infty$. Show that the function graph $(x, g(x))$ has box-counting dimension 1.*

Exercise 3.8. *Give an example of two homeomorphic sets E, F with different box-counting dimension. (This shows that the box-counting dimension is not a topological invariant.)*

Exercise 3.9. *Generalise Proposition 3.9 to Hölder functions: If $f : F \to \mathbb{R}^d$ satisfies $|f(x) - f(y)| \leq c|x-y|^\alpha$ for some $0 < \alpha \leq 1$ then $\dim_B f(F) \leq (1/\alpha)\dim_B F$, where $\dim_B$ can be the upper and lower box-counting dimension, respectively.*

Exercise 3.10. *Construct a set $F \subset \mathbb{R}$ for which $\underline{\dim}_B F < \overline{\dim}_B F$.*

3.2 Hausdorff dimension and Hausdorff measure

3.2.1 Hausdorff content and dimension

To calculate and define the Hausdorff dimension, we do not necessarily need the notion of Hausdorff measure. Here we will define the Hausdorff dimension in terms of Hausdorff content. Given any metric space (X, d), the **s-dimensional Hausdorff content** is

$$\mathcal{H}^s_\infty(X) = \inf\left\{\sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^s : X \subseteq \bigcup_{i\in\mathbb{N}} U_i\right\}$$

where the infimum is taken over any countable cover $\{U_i\}$ of X by any sets. Observe that the Hausdorff content is finite for any bounded set and $s \geq 0$ as every bounded set can be covered by a ball of some radius r_0, giving the upper bound $\mathcal{H}^s_\infty(X) \leq (2r_0)^s < \infty$.

The Hausdorff content has the property that once it reaches 0, it will stay at 0. This first occurrence of 0 content will be our definition of Hausdorff dimension.

Lemma 3.12. *Let (X, d) be a metric space. If $\mathcal{H}^s_\infty(X) = 0$ for some $s \geq 0$, then $\mathcal{H}^t_\infty(X) = 0$ for all $t > s$.*

Proof. Assume s is such that $\mathcal{H}^s_\infty(X) = 0$. Then, for all $\varepsilon > 0$, there exists a cover $\{U_i\}$ such that $\sum_i \operatorname{diam}(U_i)^s \leq \varepsilon$. In particular, choosing $\varepsilon < 1$ we must also have $\operatorname{diam}(U_i) < 1$. Then,

$$\mathcal{H}^t_\infty(X) \leq \sum_i \operatorname{diam}(U_i)^t = \sum_i \operatorname{diam}(U_i)^s \operatorname{diam}(U_i)^{t-s} \leq \sum_i \operatorname{diam}(U_i)^s \leq \varepsilon.$$

Since ε was arbitrary, our claim follows. $\square$

Definition 3.13. *The **Hausdorff dimension** of a metric space (X, d) is*

$$\dim_H X = \inf\{s > 0 : \mathcal{H}^s_\infty(X) = 0\}.$$

Equipped with the definition, we are now able to find upper bounds to the Hausdorff dimension for all the examples we have seen before. We can also remove a weakness of the box-counting dimension with the following result.

Proposition 3.14. *Let (X, d) be countable. Then, $\dim_H X = 0$.*

Proof. We can enumerate $x_i \in X$ by $\mathbb{N}$. Let $\varepsilon > 0$ and $\delta > 0$, then $U_i = B(x_i, \delta^{1/\varepsilon} 2^{-i/\varepsilon - 1})$ is a cover of X. Further,

$$\mathcal{H}^\varepsilon_\infty(X) \leq \sum_i \operatorname{diam}(B(x_i, \delta^{1/\varepsilon} 2^{-i/\varepsilon-1}))^\varepsilon = \sum_{i\in\mathbb{N}} \delta 2^{-i} = \delta$$

But $\delta > 0$ was arbitrary and so $\mathcal{H}^\varepsilon_\infty(X) = 0$ for all $\varepsilon > 0$. This shows that $\dim_H X = 0$, as required. $\square$

We can establish further properties of the Hausdorff dimension from this.

Proposition 3.15. *The Hausdorff dimension satisfies the basic properties*

1. ***Monotonicity.*** $\dim_H Y \leq \dim_H X$ *for all* $Y \subseteq X$.
2. ***Range of values.*** *If* $F \subseteq \mathbb{R}^d$, *then*

   $$0 \leq \dim_H F \leq d.$$

   If (X, d) is an arbitrary metric space, then

   $$0 \leq \dim_H X \leq \infty.$$

   Further, for all $s \in [0, d]$ there exists a compact $E \subset \mathbb{R}^d$ such that $\dim_H E = s$.
3. ***Countable stability.*** *The Hausdorff dimension is countably stable*

   $$\dim_H \bigcup_{i\in\mathbb{N}} X_i = \sup_{i\in\mathbb{N}} \dim_H X_i.$$

4. ***Open sets.*** *If $F \subseteq \mathbb{R}^d$ is non-empty and open, then* $\dim_H F = d$.
5. ***Countable sets.*** *If (X, d) is countable, then* $\dim_H X = 0$.

Proof. (1.) Monotonicity follows from the fact that $\mathcal{H}^s_\infty(Y) \leq \mathcal{H}^s_\infty(X)$ as every cover of X is a cover of Y.

(2.) The lower bound follows straight from the definition. The upper bound for general metric spaces is trivial. For sets in $\mathbb{R}^d$ it suffices to establish that $\dim_H \mathbb{R}^d \leq d$.

We will tile $\mathbb{R}^d$ by squares of various sizes. To start, let Q_i be an enumeration of d-dimensional hypercubes of sidelength $\delta > 0$. Clearly these cubes can be chosen to cover $\mathbb{R}^d$. We now divide the i-th hypercube into 2^{di} many hypercubes of sidelength $\delta \cdot 2^{-i}$ to get another countable cover of $\mathbb{R}^d$. Fix $t > d$. The diameter of a d-dimensional hypercube of sidelength l is $\sqrt{dl^2} = l\sqrt{d}$ and hence,

$$\mathcal{H}^t_\infty(\mathbb{R}^d) \leq \sum_{i \in \mathbb{N}} 2^{di} \left(\sqrt{d}\delta 2^{-i}\right)^t = \sum_{i \in \mathbb{N}} d^{t/2} \delta^t \left(\frac{2^d}{2^t}\right)^i = \delta^t C$$

for some constant C not depending on δ. But as δ was arbitrary, we have $\mathcal{H}^t_\infty(\mathbb{R}^d) = 0$ and $\dim_H \mathbb{R}^d \leq t$. Taking infima, $\dim_H \mathbb{R}^d \leq d$. Of course, this should be an equality, and we will prove so shortly. The examples of sets are postponed until the end of this section.

(3.) Let $s = \sup_i \dim_H X_i$ and choose $t > s$. Then, $\mathcal{H}^t_\infty(X_i) = 0$ for all $i \in \mathbb{N}$. Let $\varepsilon > 0$ and choose a countable cover $\{U_{i,n}\}$ of X_i such that $\sum_n \operatorname{diam}(U_{i,n})^t < \varepsilon \cdot 2^{-i}$ for all $i \in \mathbb{N}$. Then $\{U_{i,n}\}$ is a countable cover of $X = \bigcup_i X_i$ and

$$\mathcal{H}^t_\infty(X) \leq \sum_{i,n \in \mathbb{N}} \operatorname{diam}(U_{i,n})^t \leq \sum_{i \in \mathbb{N}} \varepsilon \cdot 2^{-i} = \varepsilon.$$

Thus, $\dim_H X \leq t$ and so the upper bound follows. The lower bound follows by monotonicity.

(4.) Proof postponed.

(5.) That is Proposition 3.14. □

In general, finding an upper bound to the Hausdorff dimension is simple. One only has to find a good covering. Lower bounds are harder to get to, but can be found using this fundamental lemma.

Lemma 3.16 (Mass distribution principle). *Let $E \subseteq \mathbb{R}^d$ be bounded and let μ be a strictly positive Borel measure supported on E that satisfies*

$$\mu(B(x,r)) \leq Cr^s$$

for some constant $C > 0$ and every ball $B(x,r)$. Then $\mathcal{H}^s_\infty(E) \geq \mu(E)/C$ and hence $\dim_H E \geq s$.

Proof. Let $\{U_i\}$ be a cover of E. As E bounded, we can assume each U_i is bounded. Let $x_i \in U_i$ be arbitrary and choose $r_i = \operatorname{diam}(U_i)$. Then $U_i \subseteq B(x_i, r_i)$ and

$$\mu(U_i) \leq \mu(B(x_i, r_i)) \leq Cr_i^s = C \operatorname{diam}(U_i)^s.$$

This gives

$$\sum_i \operatorname{diam}(U_i)^s \geq \sum_i \mu(U_i)/C \geq \mu(E)/C$$

and so, as the cover was arbitrary, $\mathcal{H}^s_\infty(E) \geq \mu(E)/C$ as required. □

Equipped with the mass distribution principle we can find the lower bounds to complete most of the proof of Proposition 3.15.

Proof of Theorem 3.15 (cont.). The dimension of $\mathbb{R}^d$ is d. It remains to show the lower bound. By monotonicity we can take a bounded subset of $\mathbb{R}^d$, say the unit cube $Q = [0,1]^d$. The Lebesgue measure $\mathcal{L}^d|_Q$ restricted to Q has the property that $\mathcal{L}^d(B(x,r)) \le C_d r^d$, where C_d is the volume of the d-dimensional unit ball. Therefore $\mathcal{H}^d_\infty(\mathbb{R}^d) \ge \mathcal{H}^d_\infty(Q) \ge \mathcal{L}^d(Q)/C_d = 1/C_d$. Hence $\dim_H \mathbb{R}^d \ge d$.

(Open sets) If E is open and non-empty, there exists $B(x,r) \subseteq E$. Using the Lebesgue measure $\mathcal{L}^d|_{B(x,r)}$ restricted to the ball gives $\mathcal{H}^d_\infty(E) \ge \mathcal{H}^d_\infty(B(x,r)) \ge \mathcal{L}^d(B(x,r))/C_d > 0$ by the same argument as before. Hence $\dim_H(E) \ge d$. The upper bound follows by inclusion in $\mathbb{R}^d$. □

The Hausdorff dimension is related to the box-counting dimension by being a lower bound. This can easily be established from the covering in the definition of the box-counting dimension.

Proposition 3.17. *The Hausdorff dimension is bounded above by the lower box-counting dimension, that is, for all totally bounded metric spaces (X, d),*

$$\dim_H X \le \underline{\dim}_B X.$$

Proof. Assume $t = \underline{\dim}_B X < \infty$ as otherwise there is nothing to prove. Fix $\varepsilon > 0$. By the definition of the lower box-counting dimension, there exists a sequence of scales $r_i \to 0$ as $i \to \infty$ such that $-\log N_{r_i}(X)/\log r_i \le t + \varepsilon$. Rearranging gives $N_{r_i}(X) \le r_i^{-(t+\varepsilon)}$ and thus there exists a cover of X with $N_{r_i}(X)$ balls of size r_i. Hence

$$\mathcal{H}^{t+2\varepsilon}_\infty(X) \le N_{r_i} r_i^{t+2\varepsilon} \le r_i^{t+2\varepsilon-t-\varepsilon} = r_i^\varepsilon$$

for all i. As $r_i^\varepsilon \to 0$, we have $\mathcal{H}^{t+2\varepsilon}_\infty(X) = 0$. Thus $\dim_H X \le t + 2\varepsilon$ and letting $\varepsilon \to 0$ we get the required $\dim_H X \le t = \underline{\dim}_B X$. □

As mentioned before, the trick to finding lower bounds is to find the right measure supported on the set in question. Sometimes there is an obvious choice such as a "geometrically weighted" probability measure. The Cantor measure supported on the Cantor middle-third set is a popular example in probability theory to show that a probability distribution can have uniformly convergent distribution function which is not absolutely continuous (i.e. has no Lebesgue density). This distribution function is also known as the Cantor function or devil's staircase. Here, it will help us determine the lower bound for the Hausdorff dimension of the Cantor middle-third set.

Example 3.18. The Cantor middle-third set C has Hausdorff dimension $\log 2/\log 3$.

Proof. We have already shown that $\dim_B C = \log 2/\log 3$. Hence $\dim_H C \le \dim_B C = \log 2/\log 3$. To determine a lower bound, consider the Cantor measure μ. It is constructed by giving each of the two construction intervals after removal of the middle-third half the weight of its parent interval. Giving $C_0 = [0,1]$ weight 1 we obtain a probability measure on C with the property that $\mu(I_n) = 2^{-n}$, where I_n is one of the level n construction intervals. Since these intervals are disjoint and of diameter 3^{-n}, any open ball of radius $3^{-n-1} < r < 3^{-n}$ can intersect at most two construction intervals of size 2^{-n}. Hence

$$\begin{aligned}\mu(B(x,r)) \le 2\mu(I_n) = 2^{-n+1} = 2^2 2^{-n-1} = 2^2\left(3^{\log 2/\log 3}\right)^{-n-1} \\ = 2^2\left(3^{-n-1}\right)^{\log 2/\log 3} \le 2^2 r^{\log 2/\log 3}.\end{aligned}$$

Since μ is a Borel probability measure, we can use the mass distribution principle (Proposition 3.16) and obtain $\mathcal{H}^{\log 2/\log 3}_\infty(C) \ge 2^{-2}$ giving $\dim_H C \ge \log 2/\log 3$, as required. □

Exercise 3.11. *Show that the Sierpiński gasket S satisfies* $\dim_H S = \dim_B S = \log 3/\log 2$.

3.2.2 The Hausdorff measure

As we saw in the last section, one does not need the notion of the Hausdorff measure to calculate the dimension of a set. However, the Hausdorff content has the drawback of not being an actual measure, and being highly non-additive. The refined notion of the Hausdorff measure rectifies this.

Definition 3.19. *The s-**dimensional** δ-**Hausdorff content** of a metric space (X, d) is given by*

$$\mathcal{H}^s_\delta(X) = \inf\left\{\sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^s : \bigcup_{i\in\mathbb{N}} U_i \supset X \text{ and } \operatorname{diam}(U_i) \leq \delta\right\}$$

*where the infimum is taken over all countable covers. The s-**dimensional Hausdorff measure** of X is the limit*

$$\mathcal{H}^s(X) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(X).$$

Remark 3.20. *We first remark that the limit in the Hausdorff measure is well-defined but may be infinite; any δ cover is also a δ' cover for $\delta < \delta'$ and thus $\mathcal{H}^s_\delta$ is non-decreasing in $\delta \to 0$. Further, the Hausdorff content is the least value that can be obtained for any δ, that is, $\mathcal{H}^s_\infty = \inf_{\delta>0} \mathcal{H}^s_\delta$.*

Note that the Hausdorff measure is well-defined for any subset of a metric space. As such it defines a set-valued function $\mathcal{H}^s : \mathcal{P}(X) \to [0, \infty]$. To use its full power we need to establish that it is a bona fide measure and will do so by first checking that the Hausdorff measure is a metric outer measure.

Proposition 3.21. *Let (X, d) be a metric space. The set function $\mathcal{H}^s : \mathcal{P}(X) \to [0, \infty]$ satisfies*

1. *$\mathcal{H}^s(\varnothing) = 0$,*
2. *$\mathcal{H}^s(Y) \leq \mathcal{H}^s(Z)$ for all $Y \subseteq Z \subseteq X$,*
3. *$\mathcal{H}^s(\bigcup_{i\in\mathbb{N}} X_i) \leq \sum_{i\in\mathbb{N}} \mathcal{H}^s(X_i)$,*
4. *$\inf_{y\in Y, z\in Z} d(y, z) > 0 \Rightarrow \mathcal{H}^s(Y \cup Z) = \mathcal{H}^s(Y) + \mathcal{H}^s(Z)$.*

Therefore, $\mathcal{H}^s$ is a metric outer measure.

Proof. The first property is trivially satisfied, whereas properties 2. and 3. follow from the fact that a cover of Z (the union of covers of X_i) is also a cover of Y ($\bigcup X_i$).

The last property follows from noting that $\delta = \inf_{y\in Y, z\in Z} d(y, z) > 0$ and thus, considering $\delta' < \delta/2$ coverings, we get $\mathcal{H}^s_{\delta'}(Y \cup Z) = \mathcal{H}^s_{\delta'}(Y) + \mathcal{H}^s_{\delta'}(Z)$ as no δ' covering set can intersect both Y and Z. The property for the Hausdorff measure follows upon taking limits. □

Equipped with this we can now use the following standard theorem that we will state without proof.

Theorem 3.22. *Let μ be a metric outer measure. Then all Borel sets are μ-measurable, i.e. $(X, \mathscr{B}(X), \mu)$ is a measure space.*

Therefore the Hausdorff measure is indeed a measure, where the measurable subsets include all Borel sets.

Definition 3.23. *Let $\mathcal{H}^s$ be the Hausdorff measure on (X,d). A set $Y \subseteq X$ is called $\mathcal{H}^s$-**measurable** if*

$$\mathcal{H}^s(Z) = \mathcal{H}^s(Z \cap Y) + \mathcal{H}^s(Z \cap Y^c)$$

for all $Z \subseteq X$.

We will now link the concepts of contents with that of the measure. An easy consequence of the definition is that

$$\mathcal{H}^s(X) \geq \mathcal{H}^s_\delta(X) \geq \mathcal{H}^s_\infty(X).$$

Therefore, finding a lower bound on the content, gives a lower bound on the Hausdorff measure. Recalling the mass distribution principle, we now also have a method of determining a lower bound of the Hausdorff measure. Further, a set is Hausdorff content null if and only if it is Hausdorff measure null.

Proposition 3.24. *Let (X,d) be a metric space, then*

$$\mathcal{H}^s(X) = 0 \quad \iff \quad \mathcal{H}^s_\infty(X) = 0.$$

Proof. The $\Rightarrow$ implication follows from the Hausdorff measure being an upper bound to the content. For the other direction assume $\mathcal{H}^s_\infty(X) = 0$ Then for all $\varepsilon > 0$, there exists a cover $\{U_i\}$ such that $\sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^s \leq \varepsilon$. But then $\operatorname{diam}(U_i)^s \leq \varepsilon$ and so $\{U_i\}$ is also an $\varepsilon^{1/s}$ cover. It follows that $\mathcal{H}^s_{\varepsilon^{1/s}}(X) \leq \varepsilon$. Taking a sequence $\varepsilon_i \to 0$ we also have $\varepsilon_i^{1/s} \to 0$ and so there exists a sequence $\mathcal{H}^s_{\varepsilon_i^{1/s}}(X) \leq \varepsilon_i \to 0$. Since the limit $\mathcal{H}^s_\delta(X)$ in $\delta \to 0$ must exist, we have $\mathcal{H}^s(X) = 0$ as required. $\square$

While the Hausdorff content of bounded sets is always finite, this is not true for the Hausdorff measure.

Proposition 3.25. *Let $s > 0$ and assume $\mathcal{H}^s(X) > 0$. Then, $\mathcal{H}^t(X) = \infty$ for all $t < s$. Equivalently, if $\mathcal{H}^t(X) < \infty$, then $\mathcal{H}^s(X) = 0$ for all $s > t$.*

Proof. Assume $\mathcal{H}^s(X) > 0$. Thus, given any δ-cover $\{U_i\}$, we have $\sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^s \geq \mathcal{H}^s(X) > 0$. Therefore, taking infima over δ-covers,

$$\begin{aligned}\mathcal{H}^t(X) &\geq \inf \sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^t = \inf \sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^{t-s} \operatorname{diam}(U_i)^s \\ &\geq \inf \delta^{t-s} \sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^s \geq \delta^{t-s}\, \mathcal{H}^s(X).\end{aligned}$$

But $\delta^{t-s} \to \infty$ as $\delta \to 0$ and so $\mathcal{H}^t(X) = \infty$. $\square$

As a consequence of this there exists a single value s such that $\mathcal{H}^t(X) = \infty$ for $t < s$ and $\mathcal{H}^t(X) = 0$ for $t > s$. The critical value is exactly the Hausdorff dimension (why?) and the Hausdorff measure at this critical value can take any value in $[0, \infty]$.

Exercise 3.12. *Give examples of sets $E \subseteq \mathbb{R}^d$ such that*

1. *$\mathcal{H}^s(E) = \infty$, where $s = \dim_H E$. (easy)*
2. *$\mathcal{H}^s(E) = 0$, where $s = \dim_H E$. (difficult)*

Sets that have positive and finite Hausdorff measure deserve some additional attention as well as its own name.

Definition 3.26. *Let $E \subset \mathbb{R}^d$ be a compact set with $\dim_H E = s$. If $0 < \mathcal{H}^s(E) < \infty$, we call E an **s-set**.*

In fact, most of the fractal sets we have seen thus far are s-sets.

Exercise 3.13. *Show that the middle-α Cantor sets are s-sets for $s = \log 2/(\log 2 - \log(1-\alpha))$.*

We are now also ready to justify the method used in the introduction.

Exercise 3.14. *Prove that the Hausdorff measure satisfies $\mathcal{H}^s(f(E)) = c^s\, \mathcal{H}^s(E)$, where f is any similarity satisfying $|f(x) - f(y)| = c|x-y|$.*

We conclude that the Hausdorff dimension of middle-α Cantor sets is $\log 2/(\log 2 - \log(1-\alpha))$. We can alter its structure somewhat to get an example of a set that has distinct Hausdorff, lower box-counting, and upper-box-counting dimensions. Further, its components show that the lower box-counting dimension is not finitely stable.

Example 3.27. There exists sets $E = E_0 \cup E_2$, and F such that

$$0 < \dim_H E \cup F = \dim_H F < \underline{\dim}_B E_0 \cup F = \underline{\dim}_B E_0 < \overline{\dim}_B E \cup F = \overline{\dim}_B E$$

as well as

$$\underline{\dim}_B E > \max\{\underline{\dim}_B E_0, \underline{\dim}_B E_2\}.$$

Proof. Let $N_k = 10^k - 1$ for $k \in \mathbb{N}_0$, noting that it is strictly increasing with $N_0 = 0$, and consider the following construction. Let E_0 be the set of dyadic rationals of the form

$$\sum_{i=0}^{I} a_i \cdot 2^{-i},$$

where $a_i = 0$ if $N_{4k+0} \le i < N_{4k+1}$ for some $k \ge 0$ and $a_i \in \{0,1\}$ otherwise. Similarly, define E_2 to be the dyadic rationals where $a_i = 0$ if $N_{4k+2} \le i < N_{4k+3}$ for some k, and $a_i \in \{0,1\}$ otherwise. As we have shown earlier, we can restrict the scale to $r_i = 2^{-i}$ and consider covers by closed balls of radius r_i and denote their cardinality by M_{r_i}. Consider the sequence of scales r_{i_k} for $i_k = N_{4k+2}$. Then $M_{r_{i_k}} \ge 2^{N_{4k+2} - N_{4k+1}}$ and so

$$\frac{\log M_{r_{i_k}}}{-\log r_{i_k}} \ge \frac{N_{4k+2} - N_{4k+1}}{N_{4k+2}} = 1 - \frac{N_{4k+1}}{N_{4k+2}} = 1 - \frac{10^{4k+1} - 1}{10^{4k+2} - 1} \to \frac{9}{10}.$$

Thus $\overline{\dim}_B E_0 \ge \frac{9}{10}$ and a similar argument establishes $\overline{\dim}_B E_2 \ge \frac{9}{10}$. In fact, we can easily establish the following bounds:

$$\begin{aligned}
2^{N_{4k+0} - N_{4k-3}} &\le M_{2^{-i}}(E_0) \le 2^{N_{4k+0}} && \text{for} \quad N_{4k+0} \le i < N_{4k+1}, \\
2^{i - N_{4k+1} + N_{4k+0} - N_{4k-3}} &\le M_{2^{-i}}(E_0) && \text{for} \quad N_{4k+1} \le i < N_{4k+4}, \\
2^{N_{4k+2} - N_{4k-1}} &\le M_{2^{-i}}(E_2) \le 2^{N_{4k+2}} && \text{for} \quad N_{4k+2} \le i < N_{4k+3}, \\
2^{i - N_{4k+3}} 2^{N_{4k+2} - N_{4k-1}} &\le M_{2^{-i}}(E_2) && \text{for} \quad N_{4k+3} \le i < N_{4k+6}.
\end{aligned}$$

Considering the subsequence of scales r_{i_k} for $i_k = N_{4k+1} - 1$ ($i_k = N_{4k+3} - 1$ for E_2) we obtain

$$\frac{\log M_{r_{i_k}}(E_0)}{-\log r_{i_k}} \le \frac{N_{4k+0}}{N_{4_k+1} - 1} = \frac{10^{4k} - 1}{10^{4k+1} - 2} \to \frac{1}{10}$$

along the subsequence and fixing $k \in \mathbb{N}$ we get

$$\min_{N_{4k+0} \le i < N_{4k+4}} \left\{ \frac{\log M_{2^{-i}}(E_0)}{i \log 2} \right\} = \frac{\log M_{i_k}(E_0)}{i_k \log 2} \ge \frac{N_{4k+0} - N_{4k-3}}{N_{4k+1} - 1}$$
$$= \frac{10^{4k} - 10^{4k-3}}{10^{4k+1} - 2} \to 10^{-1} - 10^{-4} = \frac{999}{10000}.$$

Thus $\frac{999}{10000} \le \underline{\dim}_B E_0 \le \frac{1}{10}$ and by symmetry $\underline{\dim}_B E_2 = \underline{\dim}_B E_0$. Let $F = C_\alpha + 3$ be a translated middle-α Cantor set for α such that $\dim_H F = \dim_B F < \frac{999}{10000}$. This shows the first inequality, noting that the Hausdorff and upper box-counting dimensions are finitely stable and the Hausdorff dimension of any countable set is 0.

For the second claim we compute a lower bound to the lower box-counting dimension of E. We need to bound

$$\frac{\log M_{2^{-i}}(E)}{i \log 2} \ge \max \left\{ \frac{\log M_{2^{-i}}(E_0)}{i \log 2}, \frac{\log M_{2^{-i}}(E_2)}{i \log 2} \right\}$$

from below. We do this by using $M_{2^{-i}}(E_2) \ge 2^{i - N_{4k+3} + N_{4k+2} - N_{4k-1}}$ for $N_{4k+4} \le i < N_{4k+6}$ and $M_{2^{-i}}(E_0) \ge 2^{i - N_{4k+5} + N_{4k+4} - N_{4k+1}}$ for $N_{4k+6} \le i < N_{4k+8}$. In the former case,

$$\frac{\log M_{2^{-i}}(E)}{i \log 2} \ge \frac{i - N_{4k+3} + N_{4k+2} - N_{4k-1}}{i} \ge 1 - \frac{N_{4k+3} - N_{4k+2} + N_{4k-1}}{N_{4k+4}}$$
$$= 1 - \frac{10^{4k+3} - 1 - 10^{4k+2} + 1 + 10^{4k-1} - 1}{10^{4k+4}} \to 1 - \frac{10^3 - 10^2 - 10^{-1}}{10^4} = \frac{90999}{100000}.$$

The latter case similarly gives

$$\frac{\log M_{2^{-i}}(E)}{i \log 2} \ge \frac{i - N_{4k+5} + N_{4k+4} - N_{4k+1}}{i} \ge 1 - \frac{N_{4k+5} - N_{4k+4} + N_{4k+1}}{N_{4k+6}}$$
$$\to 1 - \frac{10^5 - 10^4 - 10^1}{10^6} = \frac{90999}{100000}.$$

Therefore $\underline{\dim}_B E \ge \frac{90999}{100000}$ and we conclude that the lower box-counting dimension is not finitely stable. $\square$

The Hausdorff dimension behaves similarly to the box-counting dimenions under Lipschitz and bi-Lipschitz maps.

Proposition 3.28. *Let $F \subseteq \mathbb{R}^d$ and assume that $f : F \to \mathbb{R}^n$ satisfies the Hölder condition,*

$$|f(x) - f(y)| \le c|x - y|^\alpha.$$

Then, $\dim_H f(F) \le (1/\alpha) \dim_H F$. *In particular, if $\alpha = 1$ and f is Lipschitz,* $\dim_H f(F) \le \dim_H F$.

Proposition 3.29. *Let $F \subseteq \mathbb{R}^d$ and $f : F \to \mathbb{R}^n$ be bi-Lipschitz. Then,* $\dim_H f(F) = \dim_H F$.

Exercise 3.15. *Prove the Hausdorff dimension bound for Hölder maps, Proposition 3.28.*

Exercise 3.16. *Prove the Hausdorff dimension is bi-Lipschitz invariant, Proposition 3.29.*

The Hausdorff measure and dimension is a very active area of research with many interesting and fundamental results only recently established. One area is focused on when the Hausdorff measure and content coincide. For these sets, the notions of content and measure are interchangeable as the following result shows.

Theorem 3.30. *Let $F \subseteq \mathbb{R}^n$ be an $\mathcal{H}^s$-measurable set such that $\mathcal{H}^s(F) = \mathcal{H}^s_\infty(F) < \infty$, where $s = \dim_H F$. Then $\mathcal{H}^s(E) = \mathcal{H}^s_\infty(E)$ for all $\mathcal{H}^s$-measurable subsets $E \subseteq F$.*

Proof. By measurability $\mathcal{H}^s(E) = \mathcal{H}^s(F) - \mathcal{H}^s(F \setminus E)$. Then,

$$\mathcal{H}^s_\infty(E) \leq \mathcal{H}^s(E) = \mathcal{H}^s(F) - \mathcal{H}^s(F \setminus E) \leq \mathcal{H}^s_\infty(F) - \mathcal{H}^s_\infty(F \setminus E) \leq \mathcal{H}^s_\infty(E)$$

as required. □

We end our initial discussion of the Hausdorff dimension by linking the Hausdorff dimension with the topological property of being *totally disconnected*. Linking dimensions to topological properties will become more important when we explore the Assouad dimensions.

Theorem 3.31. *Let $F \subseteq \mathbb{R}^d$ be such that $\dim_H F < 1$. Then F is totally disconnected.*

Proof. Assume F contains at least two distinct points x, y as otherwise the statement is trivial. Define the pinned distance function $D_y(z) = |z - y|$ and note that

$$|D_y(z_1) - D_y(z_2)| = ||z_1 - y| - |y - z_2|| \leq |z_1 - y + y - z_2| = |z_1 - z_2|$$

by the reverse triangle inequality. Hence, D_y is Lipschitz and $\dim_H D_y(F) \leq \dim_H F < 1$. In particular this means that $D_y(F) \subset \mathbb{R}^1$ has zero Lebesgue measure and its complement must therefore be dense in $\mathbb{R}^1$. Now consider any $x \in F$. If $x \neq y$, then $r = D_y(x) \in D_y(F)$. Since $\mathbb{R} \setminus D_y(F)$ is dense in $\mathbb{R}$ there exists $0 < r_0 < r$ such that $r_0 \notin D_y(F)$. Then the open ball $B^o(y, r_0)$ satisfies $\partial B^o(y, r_0) \cap F = \varnothing$ as otherwise $r_0 \in D_y(F)$. Hence $F \subset \mathbb{R}^d \setminus \partial B^o(y, r_0)$ and

$$F = (F \cap B^o(y, r_0)) \,\cup\, (F \cap (\mathbb{R}^d \setminus \overline{B^o(y, r_o)}))$$

is the union of two open disjoint sets. Hence F is not connected and since x and y were arbitrary, F is totally disconnected. □

3.3 Packing dimension

The packing dimension is defined analogously to the Hausdorff dimension with coverings replaced by packings. It is done via the packing measure, which can be considered a dual to the Hausdorff dimension. Before we delve into it we will consider the upper box-counting dimension again and modify its construction.

3.3.1 Modified box-counting dimension

In Section 3.1 we highlighted some flaws in its simplistic definition of the box-counting dimension: countable sets can have positive dimension and unbounded sets have undefined box-counting dimension. To overcome this we can modify the box-counting dimension and "force" countable stability which also extends its definition to unbounded sets.

Definition 3.32. *Let $F \subseteq \mathbb{R}^d$. The* ***modified (upper) box-counting dimension*** *is*

$$\overline{\dim}_{\mathrm{MB}} F = \inf \left\{ \sup_{i \in \mathbb{N}} \overline{\dim}_B F_i : F \subseteq \bigcup_{i \in \mathbb{N}} F_i \right\},$$

where the infimum is over all such countable covers $\{F_i\}$ with F_i non-empty and bounded.

A similar definition can be applied to the lower box-counting dimension. All previously mentioned properties of the box-counting dimension are inherited by the modified box-counting dimension, such as bi-Lipschitz invariance. The only difference is that the dimension is now countably stable. By its definition, the modified box-counting dimension is bounded above by the upper box-counting dimension. The lower bound given by the Hausdorff dimension is slightly more involved and we obtain

$$0 \le \dim_H F \le \overline{\dim}_{\mathrm{MB}} F \le \overline{\dim}_B F \le d \tag{3.2}$$

for all $F \subset \mathbb{R}^d$.

Exercise 3.17. *Show that the Hausdorff dimension is a lower bound to the modified box-counting dimension.*

There exists a useful criterion when the box-counting and modified box-counting dimension coincide.

Proposition 3.33. *Let $F \subset \mathbb{R}^d$ be compact. Suppose that for every open $V \subset \mathbb{R}^d$ that intersects F we have $\overline{\dim}_{\mathrm{B}} F \cap V = \overline{\dim}_B F$. Then, $\overline{\dim}_{\mathrm{MB}} F = \overline{\dim}_B F$.*

Before we prove this proposition we remark that the box-counting dimension is invariant under taking closures. That is, $\overline{\dim}_B F = \overline{\dim}_B \overline{F}$. This can be seen by taking covers with closed balls whence $N_r(F) = N_r(\overline{F})$. It follows that in the definition of the modified box-counting dimension the sets F_i can be replaced by closed sets.

Proof of Proposition 3.32. A weak form of the Baire category theorem states that if X is a non-empty complete metric space and it can be written as the union of closed sets then at least one of these sets has non-empty interior. Consider (F, d) with the metric inherited from $\mathbb{R}^d$. As F is closed in $\mathbb{R}^d$, the space (F, d) is complete. Let $F_i \subseteq \mathbb{R}^d$ be a countable cover of F with closed and bounded sets. Then, $F = \bigcup_i F_i \cap F$, where $F \cap F_i$ are closed. The Baire category theorem then implies that there exists an index i_0 such that $F \cap F_{i_0}$ has non-empty interior. In other words, there exists an open subset $V \subset \mathbb{R}^d$ such that $\varnothing \neq F \cap V \subseteq F_{i_0}$. Thus, as the closed cover was arbitrary,

$$\overline{\dim}_{\mathrm{MB}} F = \inf \left\{ \sup_i \overline{\dim}_B F_i : F \subseteq \bigcup_{i \in \mathbb{N}} F_i \text{ with } F_i \text{ non-empty and compact} \right\} \ge \overline{\dim}_B F$$

The opposite inequality comes from (3.2) and we are done. □

3.3.2 Packing measure and dimension

The packing dimension is defined through the packing measure in a similar way to the Hausdorff measure. Let $s \ge 0$ and let $\delta > 0$. For $F \subseteq \mathbb{R}^d$ we define

$$\mathcal{P}_\delta^s(F) = \sup \left\{ \sum_{i \in \mathbb{N}} (2r_i)^s \ : \ \{B(x_i, r_i)\}_{i \in \mathbb{N}} \text{ is a disjoint collection of balls with } x_i \in F \text{ and } r_i \le \delta \right\},$$

where the supremum is over all such countable packings. This quantity is decreasing in $\delta \to 0$ as every δ' packing is a δ packing for $\delta' < \delta$. Therefore the limit

$$\mathcal{P}_0^s(F) = \lim_{\delta \to 0} \mathcal{P}_\delta^s(F)$$

exists but may be 0 or ∞. Unfortunately, here the analogy to the Hausdorff measure breaks down and $\mathcal{P}_0^s$ is not a measure. The situation is more akin to the box-counting dimension and we define the **packing measure** to be its countably stable variant

$$\mathcal{P}^s(F) = \inf\left\{\sum_{i\in\mathbb{N}} \mathcal{P}_0^s(F_i) \ : \ F \subseteq \bigcup_{i\in\mathbb{N}} F_i\right\},$$

where the infimum is over all such decompositions F_i. The packing measure can be confirmed to be a Borel measure on $\mathbb{R}^d$.

Exercise 3.18. *Show that $\mathcal{P}_0^s$ is not a measure and that the modification is indeed necessary.*

Exercise 3.19. *Show that $\mathcal{P}^s(F)$ is a Borel measure.*

The **packing dimension** is the critical value

$$\dim_P F = \sup\{s \geq 0 : \mathcal{P}^s(F) = \infty\} = \inf\{s \geq 0 \ : \ \mathcal{P}^s(F) = 0\}.$$

The packing dimension is nicely behaved: it is monotone, countably stable, and takes values in $[0, d]$ for subsets of $\mathbb{R}^d$. We will investigate its relation to our other dimensions by revealing a surprising fact! The packing dimension and modified box-counting dimension coincide for subsets of $\mathbb{R}^d$.

Proposition 3.34. *Let $F \subseteq \mathbb{R}^d$. Then, $\dim_P F = \overline{\dim}_{\mathrm{MB}} F$.*

Proof. We first show that the packing dimension is bounded above by the upper box-counting dimension for bounded subsets of $\mathbb{R}^d$. If $\dim_P F = 0$ we are done. Thus, choose t, s with $0 < t < s < \dim_P F$. By definition $\mathcal{P}^s(F) = \infty$ and so $\mathcal{P}_0^s(F) = \mathcal{P}_\delta^s(F) = \infty$ for any $\delta > 0$. Let $0 < \delta \leq 1$. There exists disjoint balls $B(x_i, r_i)$ with $r_i \leq \delta$ such that $\sum_{i\in\mathbb{N}}(2r_i)^s > 2^s$.

Considering the sizes of these balls, write n_k to denote the number of balls of size $2^{-k-1} < r_i \leq 2^{-k}$. We must have

$$\sum_{k\in\mathbb{N}} n_k 2^{-sk} > 1.$$

Hence, there must be k for which $n_k > 2^{tk}(2^{s-t} - 1)$ as otherwise the sum above gives

$$\sum_{k\in\mathbb{N}} n_k 2^{-sk} \leq \sum_{k\in\mathbb{N}} (2^{s-t} - 1)2^{tk-sk} = 1.$$

We conclude that any packing includes at least $n_k \geq C2^{tk}$ balls of size comparable to $2^{-k} \leq \delta$. Since δ is arbitrary the upper box-counting dimension is bounded below by t. Further, $t < s$ was arbitrary and $\overline{\dim}_B F \geq s$ as required.

We can now show that the modified box-counting dimension coincides with the packing dimension. If $F \subseteq \bigcup_{i\in\mathbb{N}} F_i$ where each F_i is non-empty and bounded we obtain

$$\dim_P F \leq \sup_i \dim_P F_i \leq \sup_i \overline{\dim}_B F_i$$

where we used countable stability in the first inequality. This proves $\dim_P F \leq \overline{\dim}_{\mathrm{MB}} F$.

Now let $s > \dim_P F$. Then $\mathcal{P}^s(F) = 0$ and there exists some collection F_i such that $F \subseteq \bigcup_{i\in\mathbb{N}} F_i$ with $\mathcal{P}_0^s(F_i) < \infty$ for all $i \in \mathbb{N}$. Then $\mathcal{P}_\delta^s(F_i)$ is uniformly bounded for small enough $\delta > 0$ and $N_r(F_i)\delta^s$ is bounded. This shows that $\overline{\dim}_B F_i \leq s$ for each $i \in \mathbb{N}$ giving the upper bound $\overline{\dim}_{\mathrm{MB}} F \leq s$ as required. $\square$

Exercise 3.20. *Let $F \subset [0,1]$ be all real numbers that do not have the digit 5 in their decimal representation. What is its Hausdorff, packing, and box-counting dimension?*

Exercise 3.21. *Let $1 \leq k \leq n$. Denote by F be the set of real numbers in the unit interval that do not have the digits $0, 1, \ldots, k-1$ in their n-ary expansion. Find its Hausdorff, packing, and box-counting dimension.*

Exercise 3.22. *In the last exercise, what happens if the k missing digits are randomly chosen for each level in the expansion?*

Exercise 3.23. *Consider the following class of Moran sets. Let $d, n \in \mathbb{N}$, $c \in (0,1)$, and let $M_0 = [0,1]^d \subset \mathbb{R}^d$ be the d-dimensional unit cube. We define the level k construction sets $M_{k,i}$ inductively, assuming that the level $k-1$ construction sets $M_{k-1,j}$ are given. Each $M_{k-1,j}$ has n many subset $M_{k,i}$ such that these subsets are mutually disjoint, non-empty, compact, and that $\operatorname{diam}(M_{k,i}) = c \cdot \operatorname{diam}(M_{k-1,j})$.*

1. *Further assume that each $M_{k,i}$ is a similar copy of its "parent" $M_{k-1,j}$. Find the Hausdorff, packing, and box-counting dimension of their (limit) Moran set.*
2. *(difficult) Show that the assumption in part 1 can be replaced by the assumption that all $M_{k,i}$ are convex and that there exists $C > 0$ such that $\operatorname{diam} \pi(M_{k,i}) > C \operatorname{diam} M_{k,i}$ for all k, i and orthogonal projections $\pi : \mathbb{R}^d \to \mathbb{R}$.*
3. *What conditions on d, n, c have to be met so that it can be realised?*

Exercise 3.24 (difficult)**.** *Let F be the real numbers in the unit interval whose ternary expansion does not contain sequentially repeated digits, for example, $0.012012012\cdots \in F$ whereas $0.11i_1i_2\ldots \notin F$, no matter what $i_k \in \{0,1,2\}$. Find its Hausdorff and box-counting dimension.*

3.4 Assouad dimension

The Assouad dimensions are a measure of maximal and minimal complexity (or thickness) of a set. Consider the upper box counting dimension. It can be shown that the box-counting dimension can be defined by

$$\overline{\dim}_B X = \inf\left\{\alpha > 0 \;:\; (\exists C > 0)(\forall 0 < r < \operatorname{diam} X) \text{ with } N_r(X) \leq C\left(\frac{\operatorname{diam} X}{r}\right)^\alpha\right\}$$

for all totally bounded metric spaces X. The constant $\operatorname{diam} X$ in the fraction, of course does not matter and can be absorbed into the constant C, but it indicates how we can modify the definition to obtain local complexity. We replace X with balls of radius R and take the supremum over all such balls.

Definition 3.35. *The (upper)* ***Assouad dimension*** *of a metric space X is*

$$\dim_A X = \inf\left\{\alpha > 0 \;:\; (\exists C > 0)(\forall 0 < r < R < \operatorname{diam} X) \sup_{x\in X} N_r(B(x,R)) < C\left(\frac{R}{r}\right)^\alpha\right\}.$$

Its natural dual, the ***lower dimension*** *(sometimes also called lower Assouad dimension) is defined analogously by*

$$\dim_L X = \sup\left\{\alpha \geq 0 \;:\; (\exists C > 0)(\forall 0 < r < R < \operatorname{diam} X) \inf_{x\in X} N_r(B(x,R)) \geq C\left(\frac{R}{r}\right)^\alpha\right\}.$$

The Assouad dimension, and especially the lower dimension, behave in a somewhat peculiar manner. For instance, the lower dimension is far from being countably or finitely stable. The addition of a single isolated point to any set drops its dimension to 0. This also means that the lower dimension is not even monotone!

We can compare the basic properties of these new dimensions with those we have found earlier, see Table 1.

Property	$\dim_L$	$\dim_H$	$\dim_P$	$\underline{\dim}_B$	$\overline{\dim}_B$	$\dim_A$
Monotonicity	✗	✓	✓	✓	✓	✓
Finite stability	✗	✓	✓	✗	✓	✓
Countable stability	✗	✓	✓	✗	✗	✗
Lipschitz stable	✗	✓	✓	✓	✓	✗
Bi-Lipschitz invariant	✓	✓	✓	✓	✓	✓
Stable under closure	✓	✗	✗	✓	✓	✓
Open set property	✗	✓	✓	✓	✓	✓

Table 1: Basic properties of our dimensions

Exercise 3.25. *Prove the missing properties or give counterexamples, as appropriate, to fill the list. You may skip showing that the Assouad dimension is not Lipschitz stable (i.e. may increase under Lipschitz maps), this will be covered after we have developed more machinery*

3.4.1 A simple example

Recall that the set of reciprocals $X = \{1/n : \mathbb{N}\} \cup \{0\}$ is compact and countable. Its Hausdorff dimension is 0 and the lower dimension is 0 by noting that any point (apart from 0) is isolated. The box-counting dimension, however, is 1/2 and we shall see that the Assouad dimension is 1. This follows from the following observation.

Definition 3.36. *Let $F \subset \mathbb{R}^d$ be non-empty. If there exists a sequence of similarities $T_n : X \to \mathbb{R}^d$ such that $T_n(X) \cap [0,1]^d$ converges to some set $E \subseteq [0,1]^d$ with respect to the Hausdorff metric, we say that E is a* ***weak tangent*** *to F.*

Proposition 3.37. *Let $F \subseteq \mathbb{R}^d$ be non-empty. Then $\dim_A F \geq \dim_A E$ for all weak tangents E of F.*

The proof of this statement is not too difficult, but we will postpone it for the time being.

Equipped with this method of finding lower bounds, consider the balls B_k of the form $B(x,R) \cap X = [0, 1/k]$, where $R \sim 1/(2k)$. Further, let $T_k(x) = kx$ be a magnification so that $X_k = T_k(X) \cap [0,1] = \{0, k/k, k/(k+1), k/(k+2), \dots\}$. We see that the distance between any two elements $x, y \in X_k$ satisfies

$$d(x,y) \geq \frac{k}{k} - \frac{k}{k+1} = \frac{1}{k+1}.$$

But then $X_k \to [0,1]$ with respect to the Hausdorff metric, and $\dim_A X \geq \dim_A[0,1] = 1$. Since $X \subseteq \mathbb{R}$ we also have the trivial upper bound and $\dim_A X = 1$.

3.4.2 Relations to other dimensions

Fixing radius R to be the diameter of the set, we see that the Assouad dimension must be an upper bound to the upper box-counting dimension, as well as the lower dimension must be a lower bound to the lower box-counting dimension. For closed sets, we can further establish that the lower dimension is a lower bound to the Hausdorff dimension.

Proposition 3.38. *Let $F \subseteq \mathbb{R}^d$ be closed. Then $\dim_L F \leq \dim_H F$.*

Proof. We assume that the lower dimension is positive and there exist s, t such that $\dim_L F > t > s > 0$, as otherwise there is nothing to prove. First, we show that there exists a constant $c > 0$ and a collection of points x_i such that every ball $B(x_i, r)$ contains c^{-s} disjoint balls $B(x_j, cr)$ of radius cr. Recall that for every minimal cover of balls of radius r there exists a maximal centred and disjoint packing of balls of radius r with comparable cardinality M_r. By the definition of the lower dimension there exists constant C such that

$$M_{cr}(B(x_i, r)) \geq kN_{cr}(B(x_i, r)) \geq kC\left(\frac{r}{cr}\right)^t = kCc^{s-t}c^{-s}.$$

Choosing c small enough such that $c^{s-t}kC \geq 1$ proves our claim.

Now choose the initial $x_0 \in F$ arbitrarily and the initial radius to be $r_0 = \min\{\operatorname{diam} F, 1\}$. Let $\mathcal{B}_0 = B(x_0, r_0)$ and let $\mathcal{B}_n$ be the union of the disjoint balls of radius $c^n r_0$. Since $\mathcal{B}_n \subseteq \mathcal{B}_{n-1}$ and $\mathcal{B}_n$ is a compact set, there exists $F' = \bigcap_{n\in\mathbb{N}} \mathcal{B}_n$. Further, since the centres are contained in F, which is closed, and the diameters of the balls are shrinking, F' is necessarily contained in F. A simple volume lemma (that we will prove below) shows that any ball $B(x, r)$ can intersect at most a constant multiple many disjoint ball of comparable radius. Hence, giving each ball in the n-th level construction weight c^{ks} we get a Bernoulli measure with the property

$$\mu(B(x, r)) \leq k\mu(B(x_i, c^n)) = kc^{ns} \leq k' r^s$$

for $c^{n+1} \leq r < c^n$. This shows that the s-dimensional Hausdorff measure is positive and by arbitrariness of s, we get the desired conclusion. □

Exercise 3.26. *Is it necessary to assume that F is closed? Either extend the proof to non-closed subsets, or give a counterexample highlighting where exactly the proof above needs closedness.*

3.5 Topological properties

We showed earlier that a set with Hausdorff dimension less than 1 is necessarily totally disconnected. The Assouad and lower dimension are also connected to topological/metric properties, namely those of doubling and uniform perfectness.

Definition 3.39. *A metric space (X, d) is* ***doubling*** *if there exists constant $D > 0$ (the* ***doubling constant*** *of the space) such that for every $x \in X$ the ball $B(x, r)$ can be covered by at most D balls of radius $r/2$.*

Proposition 3.40. *A metric space is doubling if and only if it has finite Assouad dimension.*

Proof. Let (X, d) be a doubling metric space. Let $0 < r < R < \operatorname{diam} X$ be arbitrary and set n so that $2^{n-1} < R/r \leq 2^n$. Then $B(x, R)$ can be covered by D many balls of radius $R/2$, which can be covered by D many balls of radius $R/4$, etc., continuing inductively. Since $R/2^n \leq r$ we get

$$N_r(B(x,R)) \leq N_{R/2^n}(B(x,R)) \leq D^n = (2^n)^{\log D/\log 2} \leq D\left(\frac{R}{r}\right)^{\log D/\log 2}$$

and $\dim_A X \leq \log D/\log 2 < \infty$.

For the reverse direction, let $B(x,r)$ be given and let t be such that $\dim_A X < t < \infty$. Then $D \leq N_{r/2}(B(x,r)) \leq C(r/(r/2))^t \leq C2^t$ and X is doubling. $\square$

In fact, finding embeddings from doubling spaces into Euclidean space is a very active area of research and one where the concept of Assouad dimension first emerged.

The lower dimension quantifies the notion of a perfect space. Recall that a metric space X is perfect if it has no isolated points. In other words, given any centre $x \in X$ and ball $B(x,r)$, there exists a constant c (depending on x) such that the annulus $B(x,r) \setminus B(x,cr)$ is non-empty. This leads to the notion of a uniformly perfect space.

Definition 3.41. *Let (X,d) be a metric space. We say that X is* ***uniformly perfect*** *if there exists a universal c such that $B(x,r) \setminus B(x,cr)$ is non-empty for all $x \in X$ and $0 < r < \operatorname{diam} X$.*

Proposition 3.42. *Let (X,d) be a metric space. Then X is uniformly perfect if and only if it has positive lower dimension.*

Proof. Assume $\dim_L X > t > 0$. Then, $N_r(B(x,R)) \geq C(R/r)^t$ for all $x \in X$, and $0 < r < R < \operatorname{diam} X$. Let $B(x,R)$ be arbitrary. Set c such that $C/c^t > 2$. Then $N_{cR}(B(x,R)) \geq C(R/(cR))^t > 2$. Since the minimal cover of $B(x,R)$ with radius cR has cardinality at least 2, there must be a point $y \in B(x,R)$ such that $d(x,y) > cR$. Hence X is uniformly perfect with constant c.

For the other direction, assume X is uniformly perfect with constant c. Let $B(x_0,R)$ be arbitrary and $r < R$ be given. Since X is uniformly perfect there exists x_1 with $d(x_0,x_1) > cR$. Now consider the uniformly perfect condition applied to radius $cR/4$. For every point x_0 and x_1 there exists another two points x_{01} and x_{10} such that $c^2R/4 < d(x_0,x_{01}) \leq cR/4$ and $c^2R/4 < d(x_1,x_{10}) \leq cR/4$. By the triangle inequality we also have $d(x_{01},x_{10}) > cR/2$. We can continue this construction k many times to obtain 2^k points mutually separated by $c^kR/4^{k-1}$. Letting k be such that $c^kR/4^{k-1} \sim r$, we have

$$N_r(B(x,R)) \geq 2^k = (c/4)^{k\log 2/\log(c/4)} \geq C\left(\frac{R}{r}\right)^{\log 2/\log(c/4)}$$

and so $\dim_L X \geq \log 2/\log(c/4) > 0$. $\square$

3.6 Summary

We have seen several dimensions in this section, and for the most part we will be interested in compact subsets $F \subset \mathbb{R}^d$. For such sets we get the following chain of inequalities

$$\dim_L F \leq \dim_H F \leq \dim_* F \leq \overline{\dim}_B F \leq \dim_A F,$$

where $\dim_*$ stands for the lower box-counting or packing dimension. The lower box-counting and packing dimension themselves are not comparable.

4 Iterated Function Systems

We start our investigations by proving two important lemmas. The first is a simple volume argument that shows that not too many disjoint sets of a certain size can intersect with a ball, and the second is the famed Vitali covering lemma.

4.1 Two important lemmas

In fractal geometry volume arguments are common. Here we rely on the Hausdorff (or Lebesgue) measure being bona-fide measures.

Lemma 4.1 (General Volume Lemma). *Let $\Lambda_r = \{E_i\}$ be a family of countable sets of measurable subsets of $\mathbb{R}^d$ parametrised by r such that there exist $c_1, c_2 > 0$ independent of r with $\operatorname{diam}(E) \le c_1 r$ and $\mathcal{H}^d(E) \ge c_2 r^d$ for all $E \in \Lambda_r$ and $E_1 \cap E_2 = \varnothing$ for all distinct $E_1, E_2 \in \Lambda_r$. Then there exists a constant K only depending on c_1, c_2 and d such that for every ball $B(x,r) \subset \mathbb{R}^d$ the set $\Xi_{x,r} = \{E \in \Lambda_r : B(x,r) \cap E \neq \varnothing\}$ has cardinality bounded above by K.*

Proof. Fix $r > 0$. Using measurablity and disjointness of the $E_i \in \Lambda_r$, and in particular the $E_j \in \Xi_{x,r}$, we see that $\mathcal{H}^d(\Xi_{x,r}) = \sum_{j=1}^{\#\Xi_{x,r}} \mathcal{H}^d(E_j)$. Recall also that $\mathcal{H}^d(B(x,R)) = c_d R^d$ for some c_d only depending on the ambient dimension d. Further, observe that for all $E_j \in \Xi_{x,r}$ there exists $y \in E_j$ such that $|x - y| \le r$. But since the diameter of E_j is bounded, we see that all E_j are contained in $B(x, r + c_1 r) = B(x, (1 + c_1)r)$.

Putting this together with the assumptions of the lemma we get

$$\#\Xi_{x,r} \cdot c_2\, r^d \le \sum_{j=1}^{\#\Xi_{x,r}} \mathcal{H}^d(E_j) \le \mathcal{H}^d\left(\bigcup \Xi_{x,r}\right) \le \mathcal{H}^d(B(x,(1+c_1)r)) = c_d(1+c_1)^d r^d.$$

Dividing by $c_2 r^d$ then gives the required bound $\#\Xi_{x,r} \le c_d(1+c_1)^d / c_2 =: K$. $\square$

In practice, the volume lemma allows us to bound the number of disjoint construction pieces in a ball, thereby allowing us to estimate the concentration of measure that is necessary for the mass distribution principle, where we generally try to bound $\mu(B(x,r)) \le K\mu(I_r) \le CKr^s$, where I_r is a construction piece of size r.

The Vitali lemma is similar in spirit and allows us to reduce a cover by closed balls to a small disjoint collection that "almost" covers the space. We start with the simpler finite version.

Lemma 4.2 (Finite Vitali Covering Lemma). *Let $\mathcal{B} = \{B_i\}$ be a finite collection of balls in $\mathbb{R}^d$. There exists a subcollection $\mathcal{B}' = \{B_j\} \subseteq \mathcal{B}$ such that all $B(x,r) \in \mathcal{B}'$ are mutually disjoint and*

$$\bigcup \mathcal{B} \subseteq \bigcup 3\,\mathcal{B}' = \bigcup_{B(x_j,r_j) \in \mathcal{B}'} B(x_j, 3r_j).$$

Proof. The proof is constructive. Let j_1 be such that $B_{j_1} = B(x_{j_1}, r_{j_1})$ has the largest of all radii in $\mathcal{B}$ choosing arbitrarily if there is more than one. By induction we choose a disjoint collection of balls. Assuming we have found a disjoint collection of balls $B_{j_1} \cup B_{j_2} \cup \ldots B_{j_k}$, we choose $B_{j_{k+1}}$ to be the largest ball in $\mathcal{B}$ that is disjoint from $B_{j_1} \cup \cdots \cup B_{j_k}$. We terminate the process once there is no such ball left.

To show that the enlargement contains $\bigcup \mathcal{B}$, consider an arbitrary ball $B_i \in \mathcal{B}$. If $B_i \in \mathcal{B}'$ we are done, so assume the contrary. But then B_i must intersect a ball $B_j \in \mathcal{B}'$ with no smaller radius as otherwise B_i would be a member of $\mathcal{B}'$. Hence $B_i \cap B_j \neq \varnothing$ and the triangle inequality implies that $B_i \subset 3B_j$. This proves the lemma. $\square$

The proof for arbitrary collections is similar, but requires the axiom of choice (Zorn's lemma).

Lemma 4.3 (Vitali Covering Lemma)**.** *Let $\mathcal{B}$ be an arbitrary collection of balls in a metric space (X,d) with diameter uniformly bounded above. Then there exists a disjoint subcollection $\mathcal{B}'$ such that for every $B \in \mathcal{B}$ there exists $B' \in \mathcal{B}'$ with $B \subset 5B'$.*

Proof. We partition $\mathcal{B}$ by size of balls and write $\mathcal{B}_n = \{B(x,r) \in \mathcal{B} : 2^{-n} < r \leq 2^{-n+1}\}$ for $n \in \mathbb{Z}$. By the boundedness of the balls there exists $N \in \mathbb{Z}$ such that $\mathcal{B}_n = \varnothing$ for all $n < N$ and $\mathcal{B}_N \neq \varnothing$.

We define $\mathcal{B}'$ inductively. Set $\mathcal{A}_0 = \mathcal{B}_N$ and let $\mathcal{B}'_0$ be a maximal disjoint subcollection of $\mathcal{A}_0$ (this requires the Axiom of choice). Having defined $\mathcal{A}_n$ and $\mathcal{B}_n$, we define

$$\mathcal{A}_{n+1} = \{B \in \mathcal{B}_{n+1} : B \cap B' = \varnothing \text{ for all } B' \in \mathcal{B}'_0 \cup \cdots \cup \mathcal{B}'_n\}$$

and let $\mathcal{B}'_{n+1}$ be a maximal disjoint subcollection of $\mathcal{A}_{n+1}$.

Let $\mathcal{B}' = \bigcup_{n \in \mathbb{N}_0} \mathcal{B}'_n$. It remains to show that $\mathcal{B}'$ satisfies the assumptions. Clearly, by construction, $\mathcal{B}'$ is a disjoint family of balls. Consider an arbitrary ball $B \in \mathcal{B}$. There exists n such that $B \in \mathcal{B}_n$ and we may assume B is not contained in $\mathcal{B}'_n$ as otherwise there is nothing to prove. There are two cases to consider: either $B \notin \mathcal{A}_n$, in which case there exists $B' \in \mathcal{B}'_0 \cup \cdots \cup \mathcal{B}'_{n-1}$ such that $B \cap B' \neq \varnothing$ and as $\operatorname{diam} B' > \operatorname{diam} B$ we have $B \subset 3B'$. The other case happens when $B \in \mathcal{A}_n$ where B is part of the new collection of smaller balls but is not in the maximal disjoint subset and there exists $B' \in \mathcal{B}'_n$ such that $B \cap B' \neq \varnothing$. Since $\operatorname{diam} B' < 2 \operatorname{diam} B$, the triangle inequality gives $B \subset 5B'$. This proves our claim. $\square$

A close look at the last lemma reveals that if the metric space is taken to be $\mathbb{R}^d$, the maximal subcollections are finite for bounded subsets of $\mathbb{R}^d$ and countable in general. This means that the subcollection $\mathcal{B}'$ can be assumed to be countable, even if $\mathcal{B}$ is not. In fact, one only requires separability of the underlying metric space to prove that there exists such a countable subcollection $\mathcal{B}'$.

4.2 Ahlfors regular spaces

Dimension theory is often used to determine, classify, and describe "regularity" of spaces. The different notions of dimension we have encountered can vary dramatically and we can consider a set very "regular" and "nicely behaved" if all notions of dimension coincide. In fact, some authors consider a metric space to be "fractal" only if these notions do not coincide.

Our first class of sets for which we can now prove that they are "nice" are the Ahlfors regular sets.

Definition 4.4. *A metric space (X,d) is called* ***s-Ahlfors regular*** *for $s > 0$ if there exists $C > 0$ such that*

$$\frac{1}{C} r^s \leq \mathcal{H}^s(B(x,r)) \leq C r^s$$

for all centred balls $B(x,r)$ in X.

Note that these are balls contained in X. If we are considering X as a subspace of $\mathbb{R}^d$ we need to consider balls centred in X and only containing points in X, not $\mathbb{R}^d$! Therefore, in practice, the above statement is replaced by $(1/C)r^s \leq \mathcal{H}^s|_F(B(x,r)) \leq Cr^s$ for all $x \in F$. Alternatively we may write $\mathcal{H}^s(F \cap B(x,r))$ instead of $\mathcal{H}^s|_F(B(x,r))$ if the balls are centred.

We can make some initial observations. If X is a bounded subset of $\mathbb{R}^d$, we obtain

$$0 < C^{-1}(\mathrm{diam}(X)/2)^s \leq \mathcal{H}^s(X) \leq C(\mathrm{diam}(X))^s < \infty$$

and X is an s-set. This, of course, also implies that the Hausdorff dimension is s. But more is true, the lower and Assouad dimensions are also equal to s.

Proposition 4.5. *Let $F \subset \mathbb{R}^d$ be a compact s-Ahlfors regular set. Then,*

$$\dim_L F = \dim_H F = \dim_P F = \underline{\dim}_B F = \overline{\dim}_B F = \dim_A F = s.$$

Proof. Let $B(x,R)$ be an arbitrary ball centred in F of radius $0 < R < \mathrm{diam}\, F$. Let $0 < r < R$ and consider the cover by sets $\mathcal{B}_1 = \{B(y,r) : y \in B(x,R)\}$. We can apply the Vitali covering lemma to obtain a disjoint and countable subcollection $\mathcal{B}_1'$ such that $B(x,R) \subset \bigcup 5\,\mathcal{B}_1'$. Using the measure properties of the Hausdorff measure,

$$(1/C)R^s \leq \mathcal{H}^s|_F(B(x,R)) \leq \mathcal{H}^s|_F\left(\bigcup 5\,\mathcal{B}_1'\right) \leq \#\,\mathcal{B}_1' \cdot C(5r)^s.$$

This gives $\#\,\mathcal{B}_1' \geq (5^s C^2)^{-1}(R/r)^s$. Thus there exists a constant C' such that $N_r(B(x,R)\cap F) \geq C'(R/r)^s$. This immediately implies that $\dim_L F \geq s$.

Again, consider the arbitrary ball $B(x,R)$. Let $\mathcal{B}_2$ be a maximal packing of $F \cap B(x,R)$ with balls of size $r < R$. By disjointness

$$CR^s \geq \mathcal{H}^s(B(x,R)\cap F) \geq \sum_{B(y,r)\in\mathcal{B}_2} \mathcal{H}^s(B(y,r)) \geq \#\,\mathcal{B}_2 \cdot (1/C)r^s$$

and so $\#\,\mathcal{B}_2 \leq C^2(R/r)^s$. We can conclude that there exists a constant $C' > 0$ such that $N_r(B(x,R)\cap F) \leq C'(R/r)^s$ for all $x \in F$ and $0 < r < R < \mathrm{diam}\, F$. Thus, $\dim_A F \leq s$. This proves the required inequalities. $\square$

This proves, for instance, that for all smooth d-dimensional compact manifolds their "fractal dimensions" are d since their d-dimensional volume measure (Lebesgue) is Ahlfors regular.

The opposite direction is not true. For all $s > 0$ there exists compact $F \subset \mathbb{R}^d$ such that $\dim_L F = \dim_A F = s$ but F is not s-Ahlfors regular.

Exercise 4.1. *Construct such a counterexample.*

4.3 Iterated function systems

Recall that we defined invariant sets in terms of families of contractions. If $\{f_i\}$ is a finite collection of strict contractions there exists a unique set F satisfying the invariance

$$F = \bigcup_i f_i(F).$$

The collection of contractions is often referred to as an **iterated function system**. This term derives from the ability to construct F by iterating application of the functions.

Corollary 4.6 (Corollary to Hutchinson's theorem)**.** *Let $\{f_i\}_{i=1}^N$ be a finite iterated function system on the complete metric space (X,d). Let K be any compact subset of (X,d). The invariant set $F = \bigcup_i f_i(F)$ can be written as the limit*

$$F = \lim_n \bigcup_{\substack{1\leq i_j\leq N\\ 1\leq j\leq n}} f_{i_1}\circ f_{i_2}\circ\cdots\circ f_{i_n}(K)$$

with respect to the Hausdorff metric.

The proof also follows directly from Banach's fixed point theorem as the fixed point is strictly attracting in the space of compact subsets.

This leads to several ways of thinking of invariant sets. We write $\Sigma_1 = \{1, 2, \dots, N\}$ for the index set of the iterated function system and can consider all words (also referred to as codings or sequences) of length n, written as $\Sigma_n = (\Sigma_1)^n$. The collection of all finite words is the monoid (with respect to concatenation) $\Sigma_* = \{\emptyset\} \cup \bigcup_n \Sigma_n$, where $\emptyset$ is the empty word which satisfies $v\emptyset = v$ for all words v. We write $\Sigma = (\Sigma_1)^{\mathbb{N}}$ for all infinite words. We can define the surjection $\Pi : \Sigma \to F$ by picking $x_0 \in F$ arbitrarily (though we usually pick 0 if it is in the domain of all f_i) and setting $\Pi(v) = \lim_n f_{v_1} \circ \dots \circ f_{v_n}(x_0)$. This gives us the ability to code points in the attractor F by an abstract coding. This should not be a surprise, as we have used this concept to identify points in sets before.

The Moran construction of sets can also be obtained from an iterated function system. We pick a compact set K large enough such that $f_i(K) \subset K$ for all $i \in \Sigma_1$. It is not difficult to see that the sets $M_v = f_{v_1} \circ \dots \circ f_{v_{|v|}}(K)$ form a Moran structure. But as we have seen when calculating the Hausdorff dimension, it is important to be able to bound how many Moran sets of a certain level can intersect a given ball. This is achieved, for instance, by requiring the Moran sets of the same level of construction are all disjoint. These conditions play an important role and are collectively known as separation conditions.

We will start by examining self-similar sets.

4.4 Self-similar sets

We recall the definition of a self-similar set.

Definition 4.7. *An iterated function system $\{f_i\}$ is called **self-similar** if it only contains finitely many strictly contracting similarities.*

*A set $F \subset \mathbb{R}^d$ is called **self-similar** if there exists a self-similar iterated function system $\{f_i\}_i$ such that F is invariant under $\{f_i\}$.*

Note that the iterated function system is not unique. For every iterated function system $\{f_1, \dots f_n\}$ with invariant set F, the iterated function system $\{f_1, \dots, f_n, f_1 \circ \dots \circ f_n\}$ also has invariant set F but the map $f_1 \circ \dots \circ f_n$ cannot be contained in $\{f_1, \dots, f_n\}$.

This difference might be subtle but is nevertheless important. Given a set F it might be difficult to show that there exists a self-similar iterated function system for which F is invariant. Conversely, it can be difficult to prove that a given compact set F is *not* a self-similar set. For example, the set $I = \{1\} \cup [0, \frac{1}{2}] \cup [\frac{3}{4}, \frac{7}{8}] \cup \dots \cup [\frac{2^n-1}{2^n}, \frac{2^{n+1}-1}{2^{n+1}}] \cup \dots$ consisting of countable many closed intervals and their accumulation point is a self-similar set though the defining maps may not be immediately obvious.

Exercise 4.2. *Find an IFS that generates the set I.*
(Hint: Consider three similarities, where two overlap "neatly".)

Our first dimension result is an upper bound for all self-similar sets. This quantity is known as the similarity dimension and depends solely on the iterated function system.

Definition 4.8. *Let $\{f_i\}_{i=1}^N$ be a self-similar IFS with contraction ratios c_i. The **similarity dimension** of $\{f_i\}$ is the unique non-negative solution of*

$$\sum_{i=1}^{N} (c_i)^s \left(= \sum_{i=1}^{N} (f_i'(0))^s \right) = 1.$$

Since the iterated function system is not unique, we can immediately see that it cannot (always) be equal to the Hausdorff dimension. However, it is always an upper bound.

Proposition 4.9. *Let F be the self-similar set of self-similar IFS $\{f_i\}_{i=1}^N$ with similarity ratios c_i. Then,*

$$\dim_H F \le \overline{\dim}_B F \le s,$$

where s is the similarity dimension of $\{f_i\}$.

Proof. Let $K = B(0, R)$ be a ball large enough such that $f_i(K) \subset K$ for all i. We claim such a ball exists. Recall that all maps are strict contractions and there exists $c_{\max}$ such that $c_i < c_{\max} < 1$ for all i. Let x_i be the fixed point of f_i and let $r_{\max} = \max_i |x_i|$, we prove that choosing $R > r_{\max}(1 + c_{\max})/(1 - c_{\max})$ is sufficient. Consider $y \in K$. Then $|f_i(y) - f_i(x_i)| = c_i|y - x_i|$ and

$$\begin{aligned}
f_i(K) &\subseteq B(x_i, c_{\max}(R + |x_i|)) \subseteq B(x_i, c_{\max}(R + r_{\max})) \\
&\subseteq B(0, |x_i| + c_{\max}R + c_{\max}r_{\max}) \subseteq B(0, (1 + c_{\max})r_{\max} + c_{\max}R) \\
&\subseteq B(0, (1 - c_{\max})R + c_{\max}R) = B(0, R) = K.
\end{aligned}$$

We can now construct a cover of F by considering $f_{i_1} \circ \cdots \circ f_{i_n}(K)$ for all codings of length n, that is $i = i_1 \dots i_n \in \Sigma_n$. By the multiplicativity of contraction ratios for similarities we get

$$\sum_{i \in \Sigma_n} \operatorname{diam}(f_{i_1} \circ \cdots \circ f_{i_n}(K)) = \sum_{i \in \Sigma_n} c_{i_1} c_{i_2} \dots c_{i_n} \cdot 2R = 2R \left(\sum_{i=1}^N c_i \right)^n.$$

Similarly,

$$\sum_{i \in \Sigma_n} \operatorname{diam}(f_{i_1} \circ \cdots \circ f_{i_n}(K))^s = (2R)^s \left(\sum_{i=1}^N (c_i)^s \right)^n = (2R)^s.$$

This shows that $\mathcal{H}^s_{c^n_{\max}}(F) \le (2R)^s$ and so $\mathcal{H}^s(F) \le (2R)^s$. The Hausdorff dimension result immediately follows.

For the box-counting dimension, consider the collection $\mathcal{I}_r$ of all codings $i = i_1, \dots, i_{n_i}$ such that

$$r \cdot \min c_i < c_{i_1} c_{i_2} \dots c_{i_{n_i}} \le r.$$

Using $\sum c_i^s = 1$ inductively, we see that

$$1 = \sum_{i \in \mathcal{I}_r} c^s_{i_1} c^s_{i_2} \dots c^s_{i_{n_i}} > \#\mathcal{I}_r \cdot \left(r \cdot \min_i c_i \right)^s.$$

By definition, the sets $f_{i_1} \circ \cdots \circ f_{i_{n_i}}(K)$ form a $2r$ cover of F and so $N_r(F) \le C\#\mathcal{I}_r \le C/(\min r_i^s) r^{-s}$. Hence, $\overline{\dim}_B F \le s$. ☐

The Hausdorff dimension result is, of course, trivial knowing the box-counting result. However, it is kept for illustrative purposes.

From our discussion it is clear that this dimension estimate may not be optimal. Consider the simple IFS $f_1(x) = x/3, f_2(x) = x/3 + 2/3$ that generates the Cantor middle third set. The associated similarity dimension is the solution of $\sum (c_i)^s = 1$ which gives $2(1/3)^s = 1 \Rightarrow s = \log 2/\log 3$, which we know to be correct. However, the IFS consisting of f_1, f_2 and $f_3(x) = x/9$ also generates the Cantor middle third set but has similarity dimension

$2(1/3)^s + (1/9)^s = 1 \Rightarrow (1/3)^{2s} + 2(1/3)^s = 1 \Rightarrow 1/3^s = \sqrt{2} - 1$ which gives $s = \log(\sqrt{2} - 1)/\log(1/3) \approx 0.802 > \log 2/\log 3$. This means we will need to find a way to justify that an IFS is the most optimal we can choose. This is done with separation conditions that are introduced in the next section. First, some unfinished business.

In the definition of the similarity dimension we claimed that the s was unique. This claim remains to be proven

Proposition 4.10. *Given a finite self-similar IFS, its similarity dimension exists, is non-negative, and unique.*

Proof. We first consider the trivial case when the IFS consists of only one function. Then the invariant set is a singleton and $c_1^s = 1$ indeed has only solution $s = 0$ for $0 < c < 1$. Thus we may assume that $N \geq 2$. Then, $\sum_i (c_i)^s \geq N > 1$ for $s \leq 0$. Further, the first derivative with respect to s is $\sum_i \log(c_i) \cdot c_i^s < 0$ as $\log(c_i) < 0$, whereas the second derivative is $\sum_i \log^2(c_i) \cdot c_i^s > 0$. Thus $\sum_i c_i^s$ is a continuous convex function with $\sum_i c_i^s \leq N c_{\max}^s \to 0$ as $s \to \infty$. Hence, there exists a unique $s \geq 0$ such that $\sum_i c_i^s = 1$. $\square$

4.4.1 Some Separation Conditions

Many different separation conditions exist that quantify the amount of "overlap" that is allowed. The two best known ones are the strong separation condition and the open set condition.

Definition 4.11. *Let $\{f_i\}$ be an iterated function system with associated attractor F. We say that the IFS $\{f_i\}$ satisfies the **strong separation condition (SSC)** if $f_i(F) \cap f_j(F) = \varnothing$ for all $i \neq j$.*

*A set F is said to satisfy the **strong separation condition (SSC)** if it has a generating iterated functions system that satisfies the SSC.*

An example of such a set are the middle-α Cantor sets. However, many other famous shapes, like the Sierpiński gasket, do not satisfy the SSC due to overlap at the boundaries. We previously said that this overlap is "negligible", something that we will make more formal by the next condition.

Definition 4.12. *Let $\{f_i\}$ be an iterated function system with associated attractor F. If there exists an open set U such that $f_i(U) \subset U$ for all i and $f_i(U) \cap f_j(U) = \varnothing$ for all $i \neq j$, we say that $\{f_i\}$ satisfies the **open set condition (OSC)**.*

*A set F is said to satisfy the **open set condition (OSC)** if it is invariant under an IFS that satisfies the OSC.*

While it is generally possible to check whether an IFS satisfies the strong separation condition, it is much more difficult to prove that an IFS (and hence a set F) satisfies the open set condition. Often, the set to consider is "obvious". For the Sierpiński gasket, the iterated function system does not satisfy the strong separation condition as the end points of the triangles overlap. In fact, it is impossible to remove them: any IFS preserves this gasket-like structure and a connectivity argument shows that this overlap cannot be avoided.

The Sierpiński carpet, however clearly satisfies the OSC with the open set taken to be the interior of the construction triangle. But it is not the only set that satisfies this. The open set condition is also satisfied if we take U to be the open unit square.

In practise it can be quite challenging to find open sets that are disjoint. Consider again the example $I = \{1\} \cup [0, 1/2] \cup \cdots \cup [(2^n - 1)/2^n, (2^{n+1} - 1)/2^{n+1}] \cup \ldots$ which is self-similar. However, we may not take the unit interval. Rather, the only set that works for its standard IFS is the interior $\mathrm{int}(I)$.

For reasonable mappings (say Lipschitz) the open set condition is strictly weaker than the strong separation condition.

Proposition 4.13. *Let $\{f_i\}$ be an iterated function system consisting of strict Lipschitz contractions*[8] $f_i : \mathbb{R}^d \to \mathbb{R}^d$*. If $\{f_i\}$ satisfies the strong separation condition it also satisfies the open set condition.*

Proof. The proof is essentially a compactness argument. Let $\{f_i\}$ be an IFS as above, with invariant set F. Then $f_i(F) \cap f_j(F) = \varnothing$ for $i \neq j$. Since F is compact, so are the images $f_i(F)$ and there exists $\varepsilon > 0$ such that for all $i \neq j$, for all $x \in f_i(F)$, and all $y \in f_j(F)$ we have $|x - y| > \varepsilon$. We can easily prove this claim but supposing the contrary. Then there exists a sequence $(x_i, y_i) \in f_i(F) \times f_j(F)$ such that $|x_i - y_i| \to 0$. However, as the product space of two compact spaces is itself compact, there exists a subsequence that converges and $x \in f_i(F), y \in f_j(F)$ such that $x_{i_n} \to x, y_{i_n} \to y$ and hence $|x - y| = 0$. But then $f_i(F) \cap f_j(F) \neq \varnothing$, a contradiction.

Equipped with this claim we now have that the $\varepsilon/3$ neighbourhood of the images are disjoint, i.e. $[f_i(F)]_{\varepsilon/3} \cap [f_j(F)]_{\varepsilon/3} = \varnothing$. Using the Lipschitz condition on f_i, we have $\inf_{x\in F} |f_i(x) - f_i(y)| < C \inf_{x\in F} |x - y| \leq C\delta$ for some $C < 1$, $x \in F$ and $y \in [F]_\delta$. Hence, choosing $\delta > 0$ small enough, we have $f_i([F]_\delta) \subset [f_i(F)]_{\varepsilon/3}$ and we can take U to be the open neighbourhood $[F]_\delta$ for which the images $f_i([F]_\delta)$ are disjoint. □

Exercise 4.3. *Show that a self-similar set F that satisfies the SSC is totally disconnected.*

We are now in the position to show that the Hausdorff and box-counting dimension of self-similar sets that satisfy the OSC coincides with the similarity dimension. We do this using the volume lemma, applied to the open set.

Theorem 4.14. *Let F be a self-similar set with IFS $\{f_i\}$. If F satisfies the open set condition, its Hausdorff and box-counting dimension coincide with the similarity dimension, i.e. the unique s such that*

$$\sum_i |f_i'(0)|^s = 1.$$

Further, F has positive Hausdorff measure $\mathcal{H}^s$.

Proof. In light of our previous results, we only need to show a lower Hausdorff dimension bound. We define an outer measure on the images of the open set U guaranteed by the open set condition. First, U can be assumed to be bounded as F is bounded and we may intersect U with a ball containing F. Further, as f_i are similarities we obtain

$$f_i(\overline{U}) = \overline{f_i(U)} \subset \overline{U}.$$

This implies that $F \subset \overline{U}$ and $\operatorname{diam} F \leq \operatorname{diam} U$. We may also assume that there are at least two maps f_i, f_j with different fixed points as otherwise the associated attractor is a singleton which clearly has dimension 0 and 0-dimensional Hausdorff measure 1 (as it is the counting measure).

We now define a Bernoulli measure μ on images $F_v = f_v(\overline{U})$ for words $v \in \Sigma_n$ by setting $\mu^*(F_v) = (c_{v_1} c_{v_2} \dots c_{v_n})^s$. The outer measure is

$$\mu(E) = \inf \left\{ \sum \mu^*(F_v) : E \cap F \subseteq \overline{\bigcup_v F_v} \right\}$$

[8]With a little care the domain can also be restricted.

where the infimum is taken over all countable collections of words.

Defining the measure in this way guarantees that $\mu(F) = 1$ and that the measure is consistent, i.e. measure is preserved taking sub-construction sets F_{vi}. More formally, we consider a **section** which is a countable collection of words $S \subset \Sigma_*$ such that every infinite word $w \in \Sigma$ has exactly one finite prefix contained in S. An example of such section are the level sets Σ_n. Then, given any finite word v and section S, the measure satisfies $\mu(F_v) = \mu(\bigcup_{w\in S} F_{vw}) \le \sum_{w\in S} \mu(F_{vw}) = (c_1 \dots c_n)^s$, where last inequality comes from the subadditivity of measures.

We now estimate $\mu(B(x,r))$ and start by defining $\Lambda_r = \{v \in \Sigma_* : \operatorname{diam} F_v < r \le \operatorname{diam} F_{v^-}\}$, where v^- is v with the last letter removed. This defines a finite section of words with associated image of U of size comparable to r. Since U is open we have $V_0 = \mathcal{L}^d(U) > 0$ and $V_v = \mathcal{L}^d(F_v) = V_0 \left(\frac{\operatorname{diam} F_v}{\operatorname{diam} U}\right)^d$. This and the observation that the diameter of F_v is $c_1 \dots c_n \cdot \operatorname{diam} F$ means we can apply the volume lemma to estimate

$$\mu(B(x,r)) \le K \max_{v\in\Lambda_r} \mu(F_v) \le K \max_{v\in\Lambda_r} (c_1 \dots c_{|v|})^s \le K/\operatorname{diam}(F)^s r^s$$

from which positive s-dimensional Hausdorff measure, and our result, follows. $\square$

Note that the argument crucially relies on not too much mass accumulating in a small space. It also begs the question of what happens when there is overlap. We can easily see that the similarity dimension is not the Hausdorff dimension if there are redundant maps. These are called "exact overlaps".

Definition 4.15. *An IFS $\{f_i\}$ is said to have **exact overlaps** if there exists two distinct words $v, w \in \Sigma_n$ such that $f_v = f_w$.*

If there are exact overlaps, the Hausdorff dimension is strictly less than the similarity dimension. This can be seen by considering the multiplicativity of the cylinder sizes.

Proposition 4.16. *Let $\{f_i\}$ be a self-similar IFS with associated attractor F and similarity dimension s. Assume that $\{f_i\}$ has exact overlaps. Then $\dim_H F < s$.*

Proof. Let $v \ne w$ be such that $f_v = f_w$. Let S be any section containing both v and w. Using multiplicativity inductively,

$$\begin{aligned} 1 = \sum_{i\in\Sigma_1} c_i^s = \sum_{k\in S} (c_{k_1} \dots c_{k_{|k|}})^s &= \sum_{k\in S\setminus\{v,w\}} (c_{k_1} \dots c_{k_{|k|}})^s + (c_{v_1} \dots c_{v_{|v|}})^s + (c_{w_1} \dots c_{w_{|w|}})^s \\ &> \sum_{k\in S\setminus\{v\}} (c_{k_1} \dots c_{k_{|k|}})^s. \end{aligned}$$

Further, F is clearly invariant under $\{f_k\}_{k\in S}$ and so is also invariant under $\{f_k\}_{k\in S\setminus\{v\}}$ which has similarity dimension strictly smaller s by the argument above. The required result follows. $\square$

A central conjecture in fractal geometry states that this is the only way we can make a dimension drop, i.e. have Hausdorff dimension strictly less than the similarity dimension.

Conjecture 4.17 (Dimension Drop Conjecture)**.** *Let $F \subset \mathbb{R}^d$ be a self-similar set. If there exists a generating iterated function system $\{f_i\}$ that does not have exact overlaps, the Hausdorff dimension of F is given by*

$$\dim_H F = \min\{d, s\},$$

where s is the similarity dimension of $\{f_i\}$.

Alternatively, it can be expressed as: "The Hausdorff dimension drops if and only if there are exact overlaps".

We will later see that this is true for almost every self-similar set (for some suitable definition of "almost"). The most recent significant progress was made by Hochman, who showed that the Dimension Drop Conjecture holds if all the defining parameters of the IFS are chosen to be algebraic numbers.

The general principle is: no overlaps – simple, overlaps – very complicated. This is perhaps best illustrated by showing that self-similar sets that satisfy the open set condition are Ahlfors regular.

Exercise 4.4. *Show that self-similar sets that satisfy the OSC are Ahlfors regular. (Hint: Show that the Bernoulli measure we constructed is comparable to the Hausdorff measure restricted to F.)*

This of course shows that all the other notions of dimension coincide with the similarity dimension. While treating overlaps in full generality is very difficult, there are methods that we can treat them with. These methods are called the *implicit theorems* as they do not assume knowledge of the dimension, but show other—more general—principles. This major achievement shows that quasi self-similar sets (and so all self similar and self-conformal sets) have coinciding Hausdorff and box-counting dimension, irrespective of overlap.

Theorem 4.18 (First implicit theorem (Falconer 1989))**.** *Let $F \in \mathbb{R}^d$ be a non-empty compact set such that there exists $c > 0$ such that for every closed ball $B(x, r)$ centred in F with radius $0 < r \leq \operatorname{diam} F$ there exists mapping $g : F \to B(x, r) \cap F$ with*

$$cr|y - z| \leq |g(y) - g(z)|.$$

Then, for $s = \dim_H F$, we get $\mathcal{H}^s(F) \leq 4^s c^{-s} < \infty$ and $\overline{\dim}_B F = \underline{\dim}_B F = \dim_H F = s$.

In particular, all quasi self-similar sets satisfy this condition.

Proof. Let $N_r(F)$ denote the maximal number of disjoint balls of radius r centred in F. For a contradiction assume that $N_r(F) > c^{-s}r^{-s}$ for some small $r \ll \operatorname{diam} F$. Then there exists $t > s$ such that $N_r(F) > c^{-t}r^{-t}$ and there are $N = N_r(F)$ disjoint balls B_i of size r centred in F. By assumption, there exists $g_i : F \to B_i \cap F$ such that $|g_i(y) - g_i(z)| > cr|y - z|$ and we can iterate the g_i to obtain a mass distribution similar to an IFS. In particular we set $\Sigma_1 = \{1, \ldots, N\}$ and note that $g_{v_1} \circ \ldots g_{v_n}(F)$ and $g_{w_1} \circ \cdots \circ g_{w_n}(F)$ for $v \neq w \in \Sigma_n$ are separated by $(cr)^n\delta$, where δ is the smallest distance between distinct B_i. Defining a measure by letting $\mu(g_{v_1} \circ \cdots \circ g_{v_n}) = N^{-n}$, a standard argument gives that any R-ball for $R \sim (cr)^n\delta$ can at most intersect one image $g_v(F)$ for word of length n. Thus

$$\mu(B(x, R)) \leq N^{-n} < (cr)^{tn} \leq CR^t,$$

for some $C > 0$. The mass distribution principle now implies $\dim_H F \geq t > s$, a contradiction.

The measure bound is left as an exercise. □

Exercise 4.5. *Finish the proof above. (Hint: Construct a cover from the packing.)*

Since all self-similar (and self-conformal) sets in $\mathbb{R}^d$ are also QSS, this shows that the box-counting and Hausdorff dimension must always agree.

Since the box-counting dimension is sometimes easier to estimate, this is quite a significant result. We can ask whether the same holds for the Assouad and lower dimension. For the latter, it is not too hard to see that it must also agree with the Hausdorff dimension. Every centred ball contains a not too distorted copy of the original set, giving the right lower estimate.

Exercise 4.6. *Write down a full proof that the lower dimension of a QSS set coincides with the Hausdorff dimension*

The Assouad dimension does not behave this nicely.

4.4.2 Fine structure of self-similar sets

Recall that the Assouad dimension is bounded below by the dimension of all "zoom-ins". We will make this slightly more formal here. Recall the definition of a weak tangent (Definition 3.36) and Proposition 3.37 stating that the Assouad dimension of a weak tangent is a lower bound to the Assouad dimension of the entire set.

Proof of Proposition 3.37. Let E be a weak tangent of $F \subseteq \mathbb{R}^d$. Assume $\dim_A F < s$. Then, for every similarity T_k we have

$$N_r(B(x,R) \cap T_k(F)) \leq C \left(\frac{R}{r}\right)^s$$

for all $0 < r < R$ where $C > 0$ is a universal constant. Since E is a weak tangent, there exists k such that $d_H(E, T_k(F) \cap X) < r/2$. But then for every $B(y,R) \cap E$ we can find $x \in T_k(F) \cap X$ such that $B(x, 2R) \supset B(y,R) \cap E$ and for every $r/2$ cover of $B(x,2R) \cap T_k(F)$ we may find an r cover of $B(y,R)$ of at most the same cardinality. So,

$$N_r(B(y,R) \cap E) \leq N_{r/2}(B(x,2R) \cap T_k(F)) \leq C \left(\frac{2R}{r/2}\right)^s = 4^s C \left(\frac{R}{r}\right)^s$$

and so $\dim_A E \leq \dim_A F$, as required. □

So we can characterise the Assouad dimension from below by its fine structure. Let $\mathcal{W}(F)$ be the collection of all weak tangents. We have

$$\dim_A F \geq \sup\{\dim_A E : E \in \mathcal{W}(F)\} \geq \sup\{\dim_H E : E \in \mathcal{W}(F)\}.$$

But what is more, is that the Assouad dimension can be fully characterised as such. We state this result without proof, which can be found in [7, Theorem 5.1.3], originally due to Käenmäki, Ojala, and Rossi [8].

Theorem 4.19. *Let $F \subseteq \mathbb{R}^d$ be closed and non-empty with $\dim_A F = s \in [0, d]$. Then there exists a compact set $E \subseteq \mathbb{R}^d$ with $\mathcal{H}^s(E) > 0$ which is a weak tangent.*

In particular, this means that the supremum above can be taken for the Hausdorff dimension as well as that the supremum is achieved!

Corollary 4.20. *Let $F \subseteq \mathbb{R}^d$ be closed and non-empty. Then,*

$$\dim_A F = \max\{\dim_H E : E \in \mathcal{W}(F)\}.$$

Equipped with this we will look at an example of an overlapping iterated function system. Let $f_1(x) = x/2, f_2(x) = x/3, f_3(x) = x/10 + 9/10$. The maps are chosen such that the first two share the same fixed point 0, whereas the last map has fixed point 1. Therefore, the compact convex hull of F is the unit interval $[0, 1]$. The iterated function system has exact overlaps since, e.g. $f_1 \circ f_2 = f_2 \circ f_1$. This, of course, means that its similarity dimension is a strict upper bound to its Hausdorff and box-counting dimension. The similarity dimension is given by the solution of $2^{-s} + 3^{-s} + 10^{-s} = 1$, which is approximately $s \approx 0.93226 \cdots < 1$.

To analyse the fine structure of this self similar set we require Dirichlet's theorem on Diophantine approximation.

Theorem 4.21 (Dirichlet's approximation theorem)**.** *For any real number α and integer $N \geq 1$ there exists integers p, q such that $1 \leq q \leq N$ and*

$$|q\alpha - p| \leq \frac{1}{N}.$$

This has the immediate consequence that if α is irrational, there exist infinitely many p, q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

Often it suffices to analyse the end points of intervals of the IFS. In this case we may even investigate just the neighbourhood of 0 and the endpoints under images of f_1 and f_2 as the map f_3 separates images and does not overlap with any other map. We see that

$$\{2^{-n}3^{-m} : n, m \in \mathbb{N}_0\} \subset F.$$

To investigate weak tangents of F near 0, where we may expect most overlap to occur, we may use the similarity $T_k(x) = 2^k$ and reference set $X = [0, 1]$. Thus

$$T_k(F) \cap X \supset \{2^{k-n}3^{-m} : n, m \in \mathbb{N}_0\} \cap [0, 1] = \{2^{-n}3^{-m} : n \in \mathbb{Z}, m \in \mathbb{N}_0, n \geq -k\} \cap [0, 1]$$

and so taking limits with respect to the Hausdorff distance, any limit (if it exists) must contain

$$E = \overline{\{2^{-n}3^{-m} : m \in \mathbb{N}_0, n \in \mathbb{Z}\}} \cap [0, 1].$$

We now show that this set is dense in $[0, 1]$. Therefore the limit exists and the weak tangent is $[0, 1]$. First, showing that E is dense in $[0, 1]$ is equivalent to showing that

$$\{-n \log 2 - m \log 3 : n \in \mathbb{Z}, m \in \mathbb{N}_0\}$$

is dense in $(-\infty, 0)$. But this follows from the Dirichlet approximation theorem since $\alpha = \log 2/ \log 3$ is irrational and there are infinitely many $p, q \in \mathbb{Z} \setminus \{0\}$ such that

$$\left|\frac{\log 2}{\log 3} - \frac{p}{q}\right| < \frac{1}{q^2} \Rightarrow |q \log 2 - p \log 3| < \frac{\log 3}{q}$$

and we can subdivide $(-\infty, 0)$ into intervals of length $\log(3)/q$ at $\pm n(q \log 2 - p \log 3)$.

We conclude that $[0, 1]$ is a weak tangent to F and so its Assouad dimension is full, i.e. $\dim_A F = 1$. Note that this is in stark contrast to the box-counting and Hausdorff dimension which always coincide and are bounded above by the similarity dimension.

4.5 The weak separation condition & regularity of quasi self-similar sets

It turns out that the "right" condition to look at is the weak separation property.

Definition 4.22. *Let $\{f_i\}$ be a self-similar IFS. We say that the IFS satisfies the* ***weak separation condition (WSC)*** *if*

$$\mathrm{Id} \notin \overline{\{f_v^{-1} \circ f_w : v, w \in \Sigma_*\} \setminus \{\mathrm{Id}\}},$$

where the closure is taken with respect to the pointwise topology (or the $\|.\|_\infty$ norm on $[0,1]^d$, which are equivalent for similarities).

This condition is equivalent to limiting overlaps. Let

$$\Lambda_r(x) = \{f_v : v \in \Sigma_*, |f_v'| \le r < |f_{v^-}'|, f_v(F) \cap B(x,r) \neq \varnothing\}.$$

Lemma 4.23. *A self-similar IFS satisfies the WSC if and only if there exists $M \in \mathbb{N}$ with*

$$\sup_{x\in\mathbb{R}} \sup_{r>0} \#\Lambda_r(x) \le M.$$

Proof. Without loss of generality we may assume $0 \in F$. First, assume that there exists no such M. That is, there exist x_i and r_i such that $\#\Lambda_{r_i}(x_i) > (2^{di})^{d+1}$. Since $f_v : \mathbb{R}^d \to \mathbb{R}^d$ is a similarity, it is defined by considering the images of $d+1$ non-collinear points. Let $e_0 = 0, e_1 = (1,0\dots,0), \dots, e_n = (0,\dots,0,1)$. Since $f \in \Lambda_{r_i}(x_i)$ and so $f_v(F) \cap B(x_i, r_i) \neq \varnothing$ and $|f_v'| \le r_i$ we have $f_v(0) \subseteq B(x_i, r_i + r_i \operatorname{diam} F)$ and hence $f_v(e_i) \subseteq B(x_i, (2+\operatorname{diam} F)r_i)$ for all i. The ball $B(x_i, (2+\operatorname{diam} F)r_i)$ sits naturally in a cube of sidelength $(4+2\operatorname{diam} F)r_i$ which we tile into 2^{di} many disjoint cubes of sidelength $2^{-i}(4+2\operatorname{diam} F)r_i$. Now any map $f_v \in \Lambda_{r_i}(x_i)$ induces a map $\tilde{f}_v : \{0,\dots,d\} \to \{1,\dots,2^{di}\}$ where $\tilde{f}_v(e_i)$ is the index of the cube that $f_v(e_i)$ gets mapped into. There are only $(2^{di})^{d+1}$ possible assignments and hence there are two maps $f_v, f_w \in \Lambda_{r_i}(x_i)$ that map each e_i into the same cube. But then $\|(f_v - f_w)|_{[0,1]^d}\|_\infty \le 2^{-i}(4+2\operatorname{diam} F)r_i$ and so $\|(f_v^{-1} \circ f_w - \mathrm{Id})|_{[0,1]^d}\|_\infty \le C2^{-i}$ for some universal C. We also note that f_v, f_w were distinct maps and so there are choices v_i, w_i for all i such that

$$0 < \|(f_v^{-1} \circ f_w - \mathrm{Id})|_{[0,1]^d}\|_\infty \to 0.$$

Showing that $\{f_i\}$ satisfies the WSC.

The other direction is left as an exercise. □

Exercise 4.7. *Complete the proof.*
Hint: The WSC implies that there are arbitrarily close maps. Iterate them to get arbitrarily many overlaps.

Before we can show that the WSC is the most appropriate condition for distinguishing dimension behaviour for self-similar sets, we need a couple more results for quasi self-similar sets.

Theorem 4.24 (Second implicit theorem)**.** *Let F be a compact subset of $\mathbb{R}^d$ and let $C > 0$. If for every set U that intersects F with* $\operatorname{diam} U < \operatorname{diam} F$ *there exists a mapping $g : U \cap F \to F$ satisfying*

$$|g(x) - g(y)| \ge C \operatorname{diam}(U)^{-1} |x - y|$$

then $\mathcal{H}^s(F) \ge C^s > 0$ and $\dim_B F = \dim_H F = s$.

Heuristic of proof: We assume for a contradiction that $\mathcal{H}^s(F) < C^s$ and aim to show that $\dim_H F \leq \overline{\dim}_B F < s$. The assumption implies the existence of finitely many U_i such that $\sum \operatorname{diam}(U_i)^s < C^s$ which cover F. Usinge the inverses of the maps g guaranteed by the theorem we obtain an iterated function system that allows us to estimate the box counting dimension as required. $\square$

Theorem 4.25. *Let F be a quasi self-similar set. Then*

$$\mathcal{H}^s(F \cap B(x,r)) \leq Cr^s$$

for all $x \in \mathbb{R}^d$ and $0 < r < \operatorname{diam} F$, and

$$\mathcal{H}^s(F \cap A) \leq C\, \mathcal{H}^s_\infty(F \cap A)$$

for all (not necessarily measurable!) $A \subset \mathbb{R}^d$.

Proof. We may assume that $\mathcal{H}^s(F) > 0$ since otherwise there is nothing to prove. This of course implies that $\mathcal{H}^s_\infty(F) > 0$. Write $C = 2 \cdot 2^{4s} D^{3s}\, \mathcal{H}^s_\infty(F)^{-1}$, where D is the distortion constant for the quasi self-similar set. To prove the first claim, suppose, for a contradiction, that there exist $x_0 \in \mathbb{R}^d$ and $r_0 > 0$ such that

$$\mathcal{H}^s(F \cap B(x_0, r_0)) > Cr_0^s. \tag{4.1}$$

Fix $n \in \mathbb{N}$ and let $\mathcal{B}_n$ be a maximal collection of pairwise disjoint closed balls of radius 2^{-n} centered in F. We have

$$2^{-2s}\, \mathcal{H}^s_\infty(F) 2^{ns} \leq \#\, \mathcal{B}_n \leq 2^s D^s 2^{ns}, \tag{4.2}$$

where the second inequality follows from the first implicit theorem and the first inequality can be seen from estimates in [9, §5]. The exact constant of the lower bound is not very important, and one could easily derive a non-optimal bound by considering a situation where no such constant exists. Then there exists a sequence of optimal covers giving $\mathcal{H}^s(F) = 0$, a contradiction.

For each $B \in \mathcal{B}_n$, let $g_B \colon F \to F \cap B$ be the guaranteed bi-Lipschitz map. It follows that each ball B in the packing $\mathcal{B}_n$ contains $g_B(F \cap B(x_0, r_0))$, a scaled copy of $F \cap B(x_0, r_0)$. Therefore, recalling (4.1), we get

$$\begin{aligned} \mathcal{H}^s(g_B(F \cap B(x_0, r_0))) &\geq D^{-s} 2^{-ns}\, \mathcal{H}^s(F \cap B(x_0, r_0)) \\ &> CD^{-s} 2^{-ns} r_0^s = 2 \cdot 2^{4s-ns} D^{2s}\, \mathcal{H}^s_\infty(F)^{-1} r_0^s \end{aligned} \tag{4.3}$$

for all $B \in \mathcal{B}_n$. Furthermore, since $\operatorname{diam}(g_B(F \cap B(x_0, r_0))) \leq D 2^{-n} \operatorname{diam}(F \cap B(x_0, r_0)) \leq D 2^{-n} 2 r_0 =: \delta_n$, we have

$$\mathcal{H}^s_{\delta_n}(g_B(F \cap B(x_0, r_0))) = \mathcal{H}^s_\infty(g_B(F \cap B(x_0, r_0))) \leq D^s 2^{-ns} 2^s r_0^s \tag{4.4}$$

for all $B \in \mathcal{B}_n$.

Now (4.3) and (4.2) imply

$$\sum_{B \in \mathcal{B}_n} \mathcal{H}^s(g_B(F \cap B(x_0, r_0))) \geq \#\, \mathcal{B}_n\, 2^{4s-ns+1} D^{2s}\, \mathcal{H}^s_\infty(F)^{-1} r_0^s \geq 2 \cdot 2^{2s} D^{2s} r_0^s \tag{4.5}$$

and (4.4) and (4.2) give

$$\sum_{B \in \mathcal{B}_n} \mathcal{H}^s_{\delta_n}(g_B(F \cap B(x_0, r_0))) \leq \#\, \mathcal{B}_n\, D^s 2^{-ns} 2^s r_0^s \leq 2^{2s} D^{2s} r_0^s. \tag{4.6}$$

Since, by the fact that the sets $g_B(F \cap B(x_0, r_0))$ are $\mathcal{H}^s$-measurable and (4.5),

$$\begin{aligned}\mathcal{H}^s(F) &= \mathcal{H}^s\Bigg(F \setminus \bigcup_{B \in \mathcal{B}_n} g_B(F \cap B(x_0, r_0))\Bigg) + \sum_{B \in \mathcal{B}_n} \mathcal{H}^s(g_B(F \cap B(x_0, r_0))) \\ &\geq \mathcal{H}^s\Bigg(F \setminus \bigcup_{B \in \mathcal{B}_n} g_B(F \cap B(x_0, r_0))\Bigg) + 2 \cdot 2^{2s} D^{2s} r_0^s\end{aligned}$$

and, by (4.6),

$$\begin{aligned}\mathcal{H}^s_{\delta_n}(F) &\leq \mathcal{H}^s_{\delta_n}\Bigg(F \setminus \bigcup_{B \in \mathcal{B}_n} g_B(F \cap B(x_0, r_0))\Bigg) + \sum_{B \in \mathcal{B}_n} \mathcal{H}^s_{\delta_n}(g_B(F \cap B(x_0, r_0))) \\ &\leq \mathcal{H}^s\Bigg(F \setminus \bigcup_{B \in \mathcal{B}_n} g_B(F \cap B(x_0, r_0))\Bigg) + 2^{2s} D^{2s} r_0^s,\end{aligned}$$

we conclude that

$$\mathcal{H}^s(F) - \mathcal{H}^s_{\delta_n}(F) \geq 2 \cdot 2^{2s} D^{2s} r_0^s - 2^{2s} D^{2s} r_0^s = 2^{2s} D^{2s} r_0^s > 0.$$

This is a contradiction since the lower bound is independent of n.

To show the second claim, let $A \subset \mathbb{R}^d$ and fix $\varepsilon > 0$. Choose a countable collection $\{B(x_i, r_i)\}_i$ of balls covering $F \cap A$ such that $\sum_i (2r_i)^s \leq \mathcal{H}^s_\infty(F \cap A) + \varepsilon$. Applying the first claim, we get

$$\mathcal{H}^s(F \cap A) \leq \sum_i \mathcal{H}^s(F \cap B(x_i, r_i)) \leq C \sum_i (2r_i)^s \leq C(\mathcal{H}^s_\infty(F \cap A) + \varepsilon)$$

which finishes the proof. □

Theorem 4.26. *Let F be a quasi-self-similar set. Then F is Ahlfors regular if and only if F is an s-set.*

Proof. Since any compact Ahlfors regular set is an s-set, we only need to show that any quasi-self-similar s set is Ahlfors regular. So, assuming F to be an s-set, we can apply Theorem 4.25 to obtain $\mathcal{H}^s(F \cap B(x, r)) \leq C r^s$ as required. Further, using the quasi self-similar property, any ball $B(x, r)$ contains $g(F)$, where

$$\mathcal{H}^s(F \cap B(x, r)) \geq \mathcal{H}^s(g(F)) \geq c^{-1} r^s \, \mathcal{H}^s(F).$$

This completes the proof. □

We can now show our main result for self-similar sets.

Theorem 4.27. *Let $F \subset \mathbb{R}^d$ be a self-similar set that satisfies the weak separation condition. Then F is Ahlfors regular and*

$$\dim_H F = \dim_A F = s = \lim_{r \to 0} \dim_S \Lambda_r,$$

where $\Lambda_r = \{f_v : v \in \Sigma_* \text{ and } |f'_v| \leq r < |f'_{v^-}|\}$.

Further, if the set does not satisfy the weak separation condition, then $\dim_A F \geq 1$.

We note that the expression $\dim_S \Lambda_r$, that is, the similarity dimension of the IFS given by maps of contraction comparable to r, is always an upper bound to the dimension. Hence, taking r small, this quantity is monotone and the limit exists. In the special case that the original IFS satisfies the OSC, it is constant. As an immediate corollary we get a nice dichotomy in $\mathbb{R}$.

Corollary 4.28. *Let $F \subset \mathbb{R}$ be self-similar with $0 < \dim_H F < 1$. Then the following are equivalent:*

1. *F satisfies the WSP.*
2. *F is Ahlfors regular.*
3. *F has positive Hausdorff measure.*
4. *$\dim_A F = \dim_H F$.*

Proof of Theorem 4.27. We first establish that in the WSC case the Hausdorff measure is positive by using the second implicit theorem. (Heuristics:) Let $U \subseteq \mathbb{R}^d$. Without loss of generality we may assume $U = B(x,r)$ to be a ball (why?). Since the IFS satisfies the weak separation condition, there are a finite number of maps $f_{v_1}, \ldots, f_{v_k}$ whose images of F intersect with the ball and which are of size comparable to r. Further, these maps are uniformly separated in $\|.\|_\infty$ norm. Hence we can construct a map that separates points as required.

Exercise 4.8. *The details for this argument, and the dimension bound is left as an exercise.*

Now assume that the WSC is not satisfied. For simplicity we assume that $F \subset \mathbb{R}$. The higher dimensional case is similar, though technically more challenging. We may now assume without loss of generality that the compact convex hull of F is $[0,1]$. The WSC implies that for all $\varepsilon > 0$ there are $v, w \in \Sigma_*$ such that

$$0 < \|(f_v^{-1} \circ f_w - \mathrm{Id})|_{[0,1]}\|_\infty < \varepsilon.$$

Using the mean value theorem we get

$$0 < \sup_{x\in[0,1]} |f_v(x) - f_w(x)| < \max\{|f_v'|, |f_w'|\} \cdot \varepsilon.$$

For similarities this can be improved to the statement:

There exists $C > 0$ such that for all $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ and $v, w \in \Sigma_*$ with

$$C \max\{|f_v'|, |f_w'|\} \cdot \delta < \sup_{x\in[0,1]} |f_v(x) - f_w(x)| < \min\{|f_v'|, |f_w'|\} \cdot \delta.$$

Again, the proof is left as an exercise.

Exercise 4.9. *Prove the claim for similarities.*
Hint: Use the fact that $f_v - f_w$ is itself a similarity

Using this claim, we can construct a weak tangent that is of dimension 1. The full argument is fairly technical but relies on the following construction. Fix $\varepsilon > 0$ and let $\delta \le \varepsilon$ and $v_1, w_1 \in \Sigma_*$ be as given by the claim. Let 0^k be the word of length k consisting just of the letter[9] 0. Note that

$$C|f_{v_1}'| \cdot \delta < |f_{v_1} \circ f_{0^k}(0) - f_{w_1} \circ f_{0^k}(0)| < |f_{v_1}'| \cdot \delta$$

[9]the letter, or even word, chosen is (almost) arbitrary.

and thus we can choose $k = k_1$ such that $|f'_{v_1}| \cdot \delta \sim |f'_{v_1}||f'_{0^{k_1}}|\varepsilon$. That is, we choose a dummy word 0^k such that the perturbation between the words $v_1 0^{k_1}$ and $w_1 0^{k_1}$ is comparable to ε times its contraction rate. We now proceed by induction to construct further words v_n, w_n. These are chosen using the claim, letting the new δ be much smaller than the previous perturbation. Having defined all words up to $n-1$, we get

$$C|f'_{v_n}| \cdot \delta < |f_{v_n} \circ f_{v_{n-1}} \circ f_{0^{k_{n-1}}} \circ \cdots \circ f_{v_1} \circ f_{0^{k_1}}(0) - f_{w_n} \circ \ldots f_{0^{k_1}}(0)| < |f'_{v_1}| \cdot \delta,$$

where δ is much smaller than the prior δ times the contraction rate of the previous word $v_{n-1}0^{k-1} \ldots 0^{k_1}$. We insert another dummy word 0^{k_n} such that $|(f_{v_n} \circ f_{0^{k_n}} \circ f_{v_{n-1}} \circ f_{0^{k_{n-1}}} \circ \cdots \circ f_{v_1} \circ f_{0^{k_1}})'| \cdot \varepsilon \sim |f'_{v_n}| \cdot \delta$. This construction leads to 2^n distinct words $x_n 0^{k_n} x_{n-1} \ldots 0^{k_1}$, where $x_i \in \{v_i, w_i\}$, each of which is translated from the other by a factor of ε times the contraction rate of the entire word. However, since the derivatives associated with each word are comparable, we obtain 2^n images of 0 which are separated (up to a uniform constant) by εr_n for some scale $r_n > 0$ depending on the iteration steps n. Since n is arbitrary we can "fill out" the space locally and obtain $[0,1]$ as a weak tangent. □

4.6 Exercises

Exercise 4.10. *Countable sets.*

- *Show that the set* $X = \{0\} \cup \{1/n : n \in /N\}$ *is not self-similar.*
- *Let* $X \subset \mathbb{R}^d$ *be a countably infinite set. Show that* X *cannot be self-similar.*

Exercise 4.11. *Let* $\{f_i\}$ *be an IFS consisting of bi-Lipschitz maps on* $[0,1]^d$*. Show that its invariant set* F *is either a singleton or has positive Hausdorff dimension*

Exercise 4.12. *Show that the implicit theorem fails if the set is not required to be compact.*

Exercise 4.13. *Let* $\Sigma_1 = \{0,1,2\}$ *and let* $\Sigma' \subset \Sigma$ *be all sequences such that no consecutive letters* 2 *are allowed. Let* $f_i(x) = x/3 + i/3$ *and consider the set* $F = \Pi(\Sigma')$*, where* $\Pi(v) = \lim_k f_{v_1} \circ \cdots \circ f_{v_k}(0)$*. Show that* F *is quasi self-similar and calculate its Hausdorff dimension.*

5 Self-similar multifractals

5.1 Frostman's lemma and local dimension

Recall that we already established several connections between measures and the dimension of a set. The mass distribution principle tells us that a set is of at least a certain dimension if it supports a measure that is locally not "too big". Similarly, an Ahlfors regular set carries a measure which is very regular and implies the coincidence of all dimensions we have covered.

We will expand on these links and introduce the local dimension of a measure.

Definition 5.1. *Let* μ *be a Borel measure supported on a metric space* X*. The* ***upper and lower local dimension*** *of* μ *at* $x \in X$ *are given by*

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

and

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

respectively. If the limit exists we refer to it as the ***local dimension*** *of* μ *at* x*.*

In particular, these limits capture the power law of a measure of a ball with respect to its size. Any Ahlfors regular measure, for instance, has measure $\mu(B(x,r)) \sim r^s$ and so its local dimension is s for all points in its support.

We obtain the following connections.

Proposition 5.2. *Let $F \subset \mathbb{R}^d$ be a Borel set, let μ be a finite Borel measure on $\mathbb{R}^d$ and $c \in (0,\infty)$.*

$$\text{If } \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} \le c \ (\forall x \in F) \ \text{ then } \mathcal{H}^s(F) \ge \mu(F)/c.$$

$$\text{Further, if } \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} \ge c \ (\forall x \in F) \ \text{ then } \mathcal{H}^s(F) \le 10^s \mu(\mathbb{R}^d)/c.$$

Proof. For the first implication, let $\varepsilon, \delta > 0$ and note that

$$F_\delta = \{x \in F : \mu(B(x,r)) \le (c+\varepsilon) r^s \text{ for all } r \le \delta\}$$

satisfies $F_\delta \to F$ as $\delta \to 0$ with respect to the Hausdorff metric. Let $\{U_i\}$ be a countable δ-cover of F. Since it is then also a δ-cover of F_δ, for every U_i that intersects F_δ there exists a ball B_i centred at an arbitrary point x_i in the intersection with radius $\operatorname{diam}(U_i)$. By the definition of F_δ, we have $\mu(U_i) \le \mu(B_i) \le (c+\varepsilon)\operatorname{diam}(U_i)^s$. So,

$$\mu(F_\delta) \le \sum_i \{\mu(U_i) : U_i \cap F_\delta \ne \varnothing\} \le \sum_i (c+\varepsilon)\operatorname{diam}(U_i)^s.$$

Since the cover was arbitrary, $\mu(F_\delta) \le (c+\varepsilon)\,\mathcal{H}^s_\delta(F) \le (c+\varepsilon)\,\mathcal{H}^s(F)$. Since F_δ increases monotonically in δ we obtain $\mu(F) \le (c+\varepsilon)\,\mathcal{H}^s(F)$ and the result follows by the arbitrariness of ε.

For the second implication, note that if F was unbounded and $\mathcal{H}^s(F) > 10^s/c\mu(\mathbb{R}^d)$ there must exist a bounded subset F' such that $\mathcal{H}^s(F) > 10^s/c\mu(\mathbb{R}^d)$. So we may assume F is bounded. We assume first that F is bounded and fix $\varepsilon, \delta > 0$. Consider the collection of balls

$$\mathcal{B} = \{B(x,r) : x \in F, 0 < r \le \delta \text{ and } \mu(B(x,r)) \ge (c-\varepsilon)r^s\}.$$

By assumption, for every $x \in F$ there are infinitely many r such that $B(x,r) \in \mathcal{B}$. We conclude that $\bigcup \mathcal{B} \supseteq F$ and apply the Vitali covering lemma to obtain a countable subcollection $\mathcal{B}'$ of disjoint balls such that $F \subseteq \bigcup 5\,\mathcal{B}'$. Therefore, $\{5B_i\}_{B_i \in \mathcal{B}'}$ is a 10δ-cover of F and

$$\mathcal{H}^s_{10\delta}(F) \le \sum_i \operatorname{diam}(5B_i)^s = 10^s \sum_{B(x_i,r_i)\in\mathcal{B}'} r_i^s \le \frac{10^s}{c-\varepsilon} \sum_i \mu(B_i) \le \frac{10^s}{c-\varepsilon}\mu(\mathbb{R}^d).$$

Since the upper bound is independent of δ and ε is arbitrary, we have $\mathcal{H}^s(F) \le 10^s/c\mu(\mathbb{R}^d)$. ☐

A simple corollary can be stated in terms of dimensions.

Corollary 5.3. *Let $F \subseteq \mathbb{R}^d$ be a Borel set and let μ be a finite Borel measure on $\mathbb{R}^d$.*

1. *If $\underline{\dim}_{\mathrm{loc}}\mu(x) \ge s$ for all $x \in F$ and $\mu(F) > 0$ then $\dim_H F \ge s$.*
2. *If $\underline{\dim}_{\mathrm{loc}}\mu(x) \le s$ for all $x \in F$ then $\dim_H F \le s$.*

With little effort 1) can be weakened even further, as we only need the property to hold in a subset $E \subset F$ with positive μ measure.

Frostman's lemma provides a converse to the mass distribution principle

Proposition 5.4 (Frostman lemma). *Let $F \subset \mathbb{R}^d$ be a Borel set and let $s > 0$. The following are equivalent:*

1. $\mathcal{H}^s(F) > 0$
2. *There is a positive Borel measure μ such that $\mu(B(x,r)) \leq r^s$ for all $x \in \mathbb{R}^d$ and $r > 0$.*

The arguments leading to the proof are very delicate and we will not cover it.

Note that the conclusion 2) to 1) is nothing but the mass distribution principle. We can state a local dimensional version of Frostman's lemma.

Corollary 5.5. *Let $F \subset \mathbb{R}^d$ be a non-empty Borel set. If $\dim_H F > s$ there exists μ with $0 < \mu(F) < \infty$ and $\underline{\dim}_{\mathrm{loc}}\mu(x) \geq s$ for all $x \in F$.*

As will turn out useful, we can "integrate" the statements in Corollary 5.3 (part (1)) and 5.5 to get a useful characterisation in terms of a potential theoretic criterion.

Definition 5.6. *Let $s \geq 0$. The **s-energy** of a measure μ on $\mathbb{R}^d$ is given by*

$$I_s(\mu) = \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|^s}.$$

Proposition 5.7. *Let $F \subset \mathbb{R}^d$.*

1. *If there exists a finite measure μ on F with $I_s(\mu) < \infty$ then $\mathcal{H}^s(F) = \infty$ and so $\dim_H F \geq s$.*
2. *If F is a Borel set with $\mathcal{H}^s(F) > 0$ then there exists a finite measure μ on F with $I_t(\mu) < \infty$ for all $t < s$.*

5.2 Invariant and self-similar measures

In the beginning of this course we used Banach's fixed point theorem to show that there exists a unique invariant set for every contracting iterated function system. This can also be achieved for measures, and the most important family of such measures are the push-forwards of Bernoulli measures on the underlying dynamics. We state, without proof, the uniqueness result for such measures.

Proposition 5.8. *Let $\{f_i\}$ be an iterated function system on $F \subset \mathbb{R}^d$ and let $\{p_i\}$ be a probability vector, i.e. $p_i > 0$ for all i and $\sum p_i = 1$. Then there exists a unique Borel probability measure μ such that*

$$\mu(E) = \sum_i p_i \cdot \mu(f_i^{-1}(E))$$

for all Borel sets $E \subseteq F$, and

$$\int g(x) d\mu(x) = \sum_i p_i \cdot \int g(f_i(x)) d\mu(x)$$

for all continuous $g : F \to \mathbb{R}$.

Further, the support of μ is F and if the iterated function system satisfies the strong separation condition the cylinder measure is

$$\mu(f_{i_1} \circ \cdots \circ f_{i_n}(F)) = \prod_{j=1}^{n} p_{i_j}$$

for all $i_1 \dots i_n \in \Sigma_n$.

We have, of course, seen these measures before when proving lower bounds to the Hausdorff dimension of sets. This begs the question on what the local dimensions of these sets are and whether all points in the attractor have the same local dimension. Clearly, any Ahlfors regular measure satisfies this and thus the "optimal" measure used in the proof of the open set condition lower bound has local dimension equal to the Hausdorff dimension.

Choosing a measure which is not "optimal" gives different behaviour. Consider the Cantor middle third set and consider the self-similar measure with probability $p_0 = p \in (0, 1/2)$ and $p_1 = 1 - p$. The local dimension at 0 is $\dim_{\mathrm{loc}} \mu(0) = \lim_n \log \mu(f_0^{(n)})/\log(1/3)^n = -\log p/\log 3$, whereas the local dimension at 1 is $\dim_{\mathrm{loc}} \mu(1) = -\log(1-p)/\log 3$. Using Birkhoff's ergodic theorem, we can easily see that for almost all $x \in F$ with coding $x_1 \dots x_n \dots$ and with respect to the Hausdorff measure,

$$\lim_n (1/n) \log \mu(B(x, 3^{-n})) = \lim_n \frac{1}{n} \sum_{i=1}^{n} \log p_{x_i} = \frac{\log p + \log(1-p)}{2}.$$

In particular, almost every point in the Cantor set with respect to the natural Cantor measure has local dimension $(\log p + \log(1-p))/2$. This also implies that $\dim_H\{x \in F : \dim_{\mathrm{loc}} \mu(x) = (\log p + \log(1-p))/2\} = \log 2/\log 3$. While one may be content knowing that almost every point has the expected local dimension, we also know that there are points which have different local dimension. Our goal is to determine how big the sets are that have a particular local dimension, which we will cover in the next section.

5.3 Multifractal Spectrum

Given a self-similar measure μ, supported on a self-similar set F with iterated function system $\{f_i\}$, we are interested in the achievable local dimensions and the size of the set of points with a specific local dimension. Let $\Lambda_\alpha = \{x \in F : \dim_{\mathrm{loc}} \mu(x) = \alpha\}$ be the **level sets** with prescribed local dimension and $A = \{\dim_{\mathrm{loc}} \mu(x) : x \in F\}$ be the **range of attainable local dimensions**. We will show that for self-similar IFS that satisfy the SSC, the range A is compact and convex and a singleton if and only if the measure is the self-similar measure is the "maximising measure" where $p_i = c_i^s$ with $s = \dim_H F$. The **multifractal spectrum** of the measure μ is the function $\phi : A \to \mathbb{R}$ given by $\alpha \mapsto \dim_H \Lambda_\alpha$.

The main goal in this section is to determine this spectrum.

We first deal with the degenerate case, where $p_i = c_i^s$. Clearly $\{p_i\}$ is a probability vector as $\sum c_i^s = 1$ (recall the coincidence of similarity dimension and Hausdorff dimension for self-similar sets with the SSC). Let $x \in F$ and write $v \in \Sigma$ for its unique coding. Given any cylinder $f_{v|_n}(F)$, where $n \in \mathbb{N}$, we see that $\mu(f_{v|_n}(F)) = p_{v_1} \dots p_{v_n} = c_{v_1}^s \dots c_{v_n}^s = \operatorname{diam}(f_{v|_n}(F))^s/\operatorname{diam}(F)$. Because of the strong separation condition the measure of a ball and its parent cylinder are related. We make use of the following fact, the proof of which is left as an exercise.

Lemma 5.9. *Let μ be a self-similar measure with associated iterated function system $\{f_i\}$ satisfying the SSC. Then, for every $r > 0$ and $x \in F$ with coding $v \in \Sigma$ there exists n such that*

$$(1/C)\mu(f_{v|_n}(F)) \le \mu(B(x,r)) \le C\mu(f_{v|_n}(F)) \quad \text{and} \quad (1/C)\operatorname{diam}(f_{v|_n}(F)) \le r \le C\operatorname{diam}(f_{v|_n}(F)),$$

where $C > 0$ is independent of r and x.

So,

$$\dim_{\mathrm{loc}} \mu(x) = \lim_{n\to\infty} \frac{\log \mu(f_{v|_n}(F))}{\log \operatorname{diam}(f_{v|_n}(F))} = \lim_{n\to\infty} s - \frac{\log \operatorname{diam}(F)}{\log \operatorname{diam}(f_{v|_n}(F))} = s$$

for all $x \in F$. Hence the multifractal spectrum is the trivial function $\phi : A = \{s\} \to \{s\}$ given by $\phi(s) = s$. Throughout the remainder, we will assume that we are not dealing with this "maximising measure" to avoid trivial cases.

It turns out that the Hausdorff dimension of the levels sets Λ_α is linked to an implicitly defined auxiliary function similar to the similarity dimension. Let $q \in \mathbb{R}$ and define $\beta(q)$ to satisfy

$$\sum_i p_i^q c_i^{\beta(q)} = 1.$$

Similar to our previous result one can show that this value $\beta(q)$ is unique and even continuous in q. We will omit details here. We also note that for $q = 0$ this equation reduces to the similarity dimension equation. Another immediate observation is that for the degenerate case $p_i = c_i^s$ we have

$$1 = \sum_i p_i^q c_i^{\beta(q)} = \sum_i c_i^{sq+\beta(q)}$$

and so $sq + \beta(q) = s$, giving $\beta(q) = s(1-q)$. In particular, it is a linear function in q with slope $-s$, intersecting the y-axis at s.

In general we cannot solve this equation for $\beta(q)$ but can differentiate implicitly to get more information. The first derivative is

$$0 = \frac{d}{dq}\sum_i p_i^q c_i^{\beta(q)} = \sum_i p_i^q c_i^{\beta(q)} \left(\log p_i + \frac{d\beta}{dq}\log c_i\right)$$

and differentiating again gives

$$0 = \sum_i p_i^q c_i^{\beta(q)} \left(\frac{d^2\beta}{dq^2}\log c_i + \left(\log p_i + \frac{d\beta}{dq}\log c_i\right)^2\right).$$

This, however means that $d\beta^2/dq^2 \geq 0$, and strictly so in the non-degenerate case. Hence β is a strictly convex function. Its Legendre transform is the multifractal spectrum that we are searching for,

$$\phi(\alpha) = \inf_{q\in\mathbb{R}}\{\beta(q) + \alpha q\}.$$

The (strict) convexity implies that there are two values $\alpha_{\min}$ and $\alpha_{\max}$, the (absolute value of the) asymptotic slopes of β as $q \to \infty$ and $q \to -\infty$, respectively. The Legendre transform is well-defined for all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, which proves the convexity and compactness of the range of attainable values. (Provided we prove that the Legendre transform is indeed the multifractal spectrum!)

What is clear is that for all such α there is a unique value q that attains the infimum and differentiating $\beta(q) + \alpha q$ and setting it equal to 0 gives $\beta'(q) = \alpha$. This gives

$$\phi(\alpha) = \alpha q + \beta(q) = -q\beta'(q) + \beta(q).$$

Rearranging the first implicit derivative gives

$$\alpha = \frac{\sum_i p_i^q c_i^\beta \log p_i}{\sum_i p_i^q c_i^\beta \log c_i}.$$

Which, in turn, give

$$\alpha_{\min} = \min_i \frac{\log p_i}{\log c_i} \quad \text{and} \quad \alpha_{\max} = \max_i \frac{\log p_i}{\log c_i}.$$

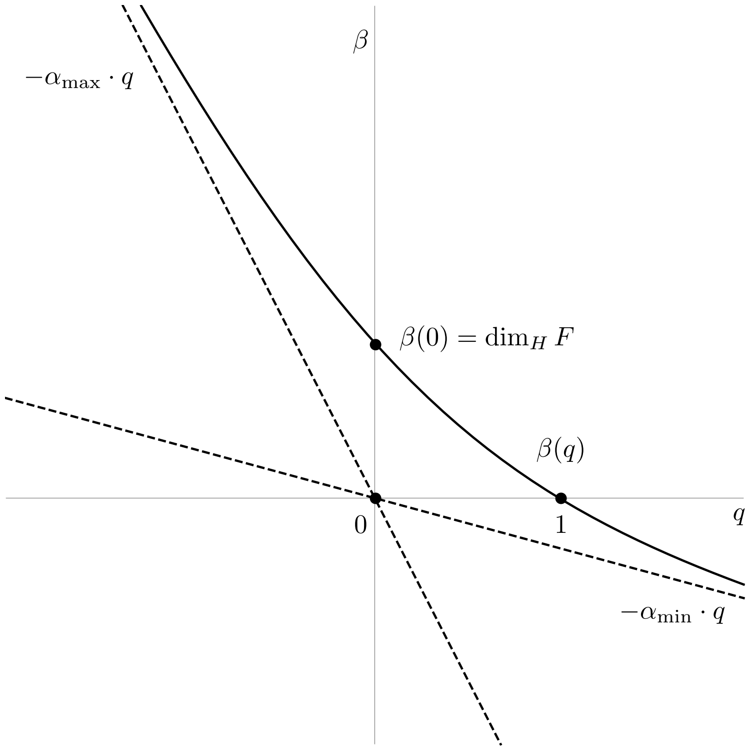


Figure 12: The function $\beta(q)$.

So what does the spectrum look like in the non-degenerate case? At $q = 0$, we have $\phi(\alpha(0)) = \dim_H \operatorname{supp} \mu$ with $\alpha(0) = \sum c_i \log p_i / \sum c_i \log c_i$. Differentiating ϕ with respect to α gives

$$\phi'(\alpha) = \alpha \frac{dq}{d\alpha} + q + \frac{d\beta}{dq}\frac{dq}{d\alpha} = q$$

and as q decreases as α increases, we see that ϕ is strictly concave with maximum at $q = 0$.

Theorem 5.10. *Let μ be a non-degenerate self-similar measure and let $\Lambda_\alpha, \alpha_{\min}, \alpha_{\max}, \beta(q)$ be as above. If $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$, then $\Lambda_\alpha = \varnothing$. If $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ then,*

$$\phi(\alpha) := \dim_H \Lambda_\alpha = \widetilde{\beta}(\alpha),$$

where $\widetilde{\beta}(\alpha)$ is the Legendre transform of $\beta(q)$.

We first provide only a partial proof, giving the upper bound.

Upper bound for the multifractal spectrum. Let $\varepsilon > 0$ and consider the collection of words Ω_k of words of length k defined by

$$\Omega_k = \{v \in \Sigma_k : \mu(f_v(F)) \geq \operatorname{diam}(f_v(F))^{\alpha+\varepsilon}\}.$$

Then, assuming $q > 0$,

$$\sum_{v \in \Omega_k} \operatorname{diam}(f_v(F))^{\beta + q\cdot(\alpha+\varepsilon)} \leq \sum_{v \in \Omega_k} \operatorname{diam}(f_v(F))^{\beta} \mu(f_v(F))^q$$

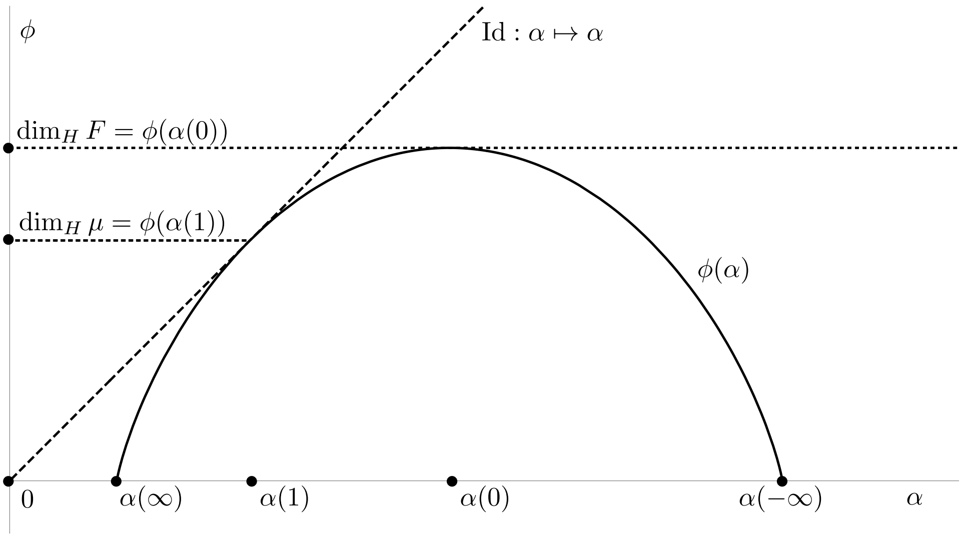


Figure 13: The multifractal spectrum $\phi(\alpha(q))$.

$$
\begin{aligned}
&\leq \sum_{v \in \Sigma_k} \operatorname{diam}(f_v(F))^\beta \mu(f_v(F))^q \\
&= \sum_{v_1, \ldots, v_k \in \Sigma_1} (c_{v_1} \ldots c_{v_k})^\beta (p_{v_1} \ldots p_{v_k})^q \\
&= \left(\sum_i p_i^q c_i^\beta \right)^k = 1.
\end{aligned}
$$

Since every $x \in F$ has a unique coding, we slightly abuse notation and write $x_k = v|_k$, where $v \in \Sigma$ is the unique $v \in \Sigma$ such that $\Pi(v) = x$. Then,

$$F_k = \{x \in F : \mu(f_{x_n}(F)) \geq \operatorname{diam}(f_{x_n}(F))^{\alpha+\varepsilon} \text{ for all } n \geq k\}$$

has the property that $F_k \subseteq \Pi(\Omega_n)$ for all $n \geq k$. Therefore, $\mathcal{H}^{\beta+q(\alpha+\varepsilon)}_{c^n_{\max}}(F_k) \leq 1$ and taking limits in n, $\dim_H F_k \leq \beta + q \cdot (\alpha + \varepsilon)$. Since any point x with local dimension α must eventually satisfy $\mu(f_{x_n}(F)) \geq \operatorname{diam}(f_{x_n}(F))^{\alpha+\varepsilon}$ (Lemma 5.9) we must have $\Lambda_\alpha \subseteq \bigcup_{k=1}^\infty F_k$. Countable stability then shows that $\dim_H \Lambda_\alpha \leq \beta(q) + q \cdot (\alpha + \varepsilon)$ for all $q > 0$. The case for $q < 0$ is similar (up to some sign changes) and for $q = 0$ we have the trivial bound $\dim_H \Lambda_\alpha \leq \dim_H \operatorname{supp} \mu$. Taking $\varepsilon \to 0$ completes the upper bound. □

5.3.1 The lower bound (optional)

The lower bound can be constructed by carefully choosing a mass distribution on sets with given asymptotic mass to size ratio. The details are long and tricky, though doable. This was how the result was originally proven in [3], which also shows that $\dim_H \Lambda_{>\alpha} = \dim_H \Lambda_{\geq\alpha} = \sup_{\alpha'>\alpha} \dim_H \Lambda_\alpha$ and a corresponding result for $\Lambda_{\leq\alpha}, \Lambda_{<\alpha}$. Here we take the somewhat shorter approach of Falconer, see [4, §17].

We first show that for sufficiently small perturbations from the dimension, the modified sum is less than unity.

Lemma 5.11. *Let $\varepsilon > 0$. Then, for sufficiently small $\delta > 0$,*

$$\mathfrak{S}(q+\delta, \beta(q) + (-\alpha+\varepsilon)\delta) < 1 \quad \textit{and} \quad \mathfrak{S}(q-\delta, \beta(q) + (\alpha+\varepsilon)\delta) < 1,$$

where

$$\mathfrak{S}(q,\beta) = \sum_i p_i^q c_i^\beta.$$

Proof. A Taylor expansion of $\beta(q)$ gives

$$\beta(q+\delta) = \beta(q) + \beta'(q)\delta + O(\delta^2) = \beta(q) - \alpha\delta + O(\delta^2) < \beta(q) + (-\alpha+\varepsilon)\delta$$

for $\delta > 0$ small enough. Observe that $\mathfrak{S}(q,\beta)$ is decreasing in β and so

$$1 = \mathfrak{S}(q+\delta, \beta(q+\delta)) > \mathfrak{S}(q+\delta, \beta(q) + (-\alpha+\varepsilon)\delta)$$

as required. The second inequality is analogous. $\square$

The proof relies on constructed a probability measure ν by repeated subdivision, using that

$$\sum_i p_i^q c_i^\beta = 1.$$

Given any word $v \in \Sigma$, we have three relevant quantities of the associated geometric cylinder. Its diameter, its μ measure, and its ν measure. They are, respectively

$$\operatorname{diam}(f_{v|_n}(F)) = c_{v_1} \dots c_{v_n} \cdot \operatorname{diam}(F), \quad \mu(f_{v|_n}(F)) = p_{v_1} \dots p_{v_n},$$

and

$$\nu(f_{v|_n}(F)) = (p_{v_1} \dots p_{v_n})^q (c_{v_1} \dots c_{v_n})^\beta = \left(\frac{\operatorname{diam}(f_{v|_n}(F))}{\operatorname{diam}(F)}\right)^\beta \mu(f_{v|_n}(F))^q. \tag{5.1}$$

This measure is a probability measure on Λ_α and has the property that $\dim_{\mathrm{loc}} \nu(x) = q\alpha + \beta(q)$ for all $x \in \Lambda$. We first prove the latter claim as it follows easily from (5.1).

$$\begin{aligned}
\frac{\log \nu(B(x,r))}{\log r} &\sim \frac{\log \nu(f_{v|_n}(F))}{\log \operatorname{diam}(f_{v|_n}(F))} \\
&= q \cdot \frac{\log \mu(f_{v|_n}(F))}{\log \operatorname{diam}(f_{v|_n}(F))} + \beta(q) \cdot \left(\frac{\log \operatorname{diam} f_{v|_n}(F))}{\log \operatorname{diam}(f_{v|_n}(F))} + \frac{\log \operatorname{diam}(F)}{\log \operatorname{diam}(f_{v|_n}(F))}\right) \\
&\to q\alpha + \beta(q)
\end{aligned}$$

for all $x \in \Lambda_\alpha$ and $q \in \mathbb{R}$.

We now also have to show that $\nu(\Lambda_\alpha) = 1$. We do this by estimating the probability that cylinders have more mass than expected for its local dimension. Fix $\varepsilon > 0$. Denote by χ_A the indicator function for the event A. Then for all $\delta > 0$,

$$\begin{aligned}
\nu\{x \in F : \mu(f_{x_k}(F)) \geq \operatorname{diam}(f_{x_k}(F))^{\alpha-\varepsilon}\} &= \nu\{x \in F : \mu(f_{x_k}(F))^\delta \operatorname{diam}(f_{x_k}(F))^{-\delta(\alpha-\varepsilon)} \geq 1\} \\
&= \int \chi_{\mu(f_{x_k}(F))^\delta \operatorname{diam}(f_{x_k}(F))^{-\delta(\alpha-\varepsilon)} \geq 1}(x) d\nu(x) \\
&\leq \int \mu(f_{x_k}(F))^\delta \operatorname{diam}(f_{x_k}(F))^{-\delta(\alpha-\varepsilon)} d\nu(x)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{v\in\Sigma_k} \mu(f_{v|_k}(F))^{\delta} \operatorname{diam}(f_{v|_k}(F))^{-\delta(\alpha-\varepsilon)} \nu(f_{v|_k}(F)) \\
&= \operatorname{diam}(F)^{-\beta} \sum_{v\in\Sigma_k} \mu(f_{v|_k}(F))^{\delta+q} \operatorname{diam}(f_{v|_k}(F))^{\beta-\delta(\alpha-\varepsilon)} \\
&= \operatorname{diam}(F)^{-\beta}\, \mathfrak{S}(q+\delta, \beta(q) + (-\alpha+\varepsilon))^k \\
&\leq \operatorname{diam}(F)^{-\beta} \gamma^k
\end{aligned}
$$

for some $\gamma < 1$ by the Lemma above. However, this is summable in k and the Borel-Cantelli implies that

$$\nu\{x \in F : \mu(f_{x_k}(F) \geq \operatorname{diam}(f_{x_k}(F)) \text{ for infinitely many } k.\} = 0.$$

Thus, ν-almost surely, $\underline{\dim}_{\text{loc}}\mu(x) \geq \alpha - \varepsilon$. Analogously, one can show that ν-almost surely $\overline{\dim}_{\text{loc}}\mu(x) \leq \alpha+\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\dim_{\text{loc}} \mu(x) = \alpha$ for ν-almost every x and therefore $\nu(\Lambda_\alpha) = \nu(F) = 1$.

This shows that there exists a probability measure ν supported on Λ_α such that $\nu(\Lambda_\alpha) = 1$ and $\dim_{\text{loc}} \nu(x) = \phi(\alpha)$. We can apply Corollary 5.3 (1) to give $\dim_H \Lambda_\alpha \geq \phi(\alpha)$ which finishes the proof[10].

5.3.2 The Hausdorff dimension of a measure

There notion of the Hausdorff dimension of a measure is related to the fact that the Hausdorff dimension is not stable under closure. While the support of a measure is necessarily closed, sets with full measure may not be. This is not surprising, as the Lebesgue measure restricted to $[0, 1]$ is full on its interior $(0, 1)$. Similarly, the support of a measure may have dimension strictly greater than its closure. Consider for instance the measure μ giving weight $1/n$ to the n-th element a_n in an enumeration of the rationals. While $\mu(\mathbb{Q}) = \mu(\mathbb{R}) = 1$, we of course have $\dim_H \mathbb{Q} = 0 < \dim_H \mathbb{R} = 1$.

This phenomenon inspires the Hausdorff dimension of a measure μ, which is the least Hausdorff dimension s such that there exists a (measurable) set E with $\mu(E) > 0$. Formally, the **Hausdorff dimension of a Borel measure** μ is

$$\dim_H \mu = \inf\{\dim_H E : \mu(E) > 0 \text{ and } E \text{ is Borel}\}.$$

Curiously, the Hausdorff dimension of a self-similar measure for which the IFS satisfies the SSC can easily be taken from the multifractal spectrum of μ. It is given at $q = 1$, that is $\dim_H \mu = \phi(\alpha(1))$.

Proposition 5.12. *Let μ be a self-similar measure such that its associated IFS satisfies the SSC. Then,*

$$\dim_H \mu = \phi(\alpha(1)) = \alpha(1) = \frac{\sum_i p_i \log p_i}{\sum_i p_i \log c_i}.$$

Proof. For $q = 1$ we have $\beta(q) = 0$ and so the measure ν in the proof of the multifractal formalism satisfies

$$\nu(f_v(F)) = (p_{v_1} \dots p_{v_n})^q (c_{v_1} \dots c_{v_n})^{\beta} = p_{v_1} \dots p_{v_n} = \mu(f_v(F))$$

for all $v \in \Sigma_n$. But then $\nu = \mu$ and $\nu(\Lambda_{\alpha(1)}) = \mu(\Lambda_{\alpha(1)}) = 1$. Hence $\dim_H \mu = \dim_H \Lambda_{\alpha(1)} = \phi(\alpha(1))$ as required. The other formulae follow from our previous results. $\square$

[10] Instead of the upper bound found earlier, we could alternatively apply Corollary 5.3(2) to give the upper bound.

5.4 Exercises

Exercise 5.1. *Prove Lemma 5.9.*

Exercise 5.2. *What is the box-counting dimension spectra, i.e.* $\varphi(\alpha) = \dim_B \Lambda_\alpha$*?*

Exercise 5.3. *Is it possible that* $\phi(\alpha_{\min}) \neq 0$ *for non-trivial, non-degenerate self-similar measures satisfying the SSC? If so, give a necessary and sufficient condition on the iterated function system such that* $\phi(\alpha_{\min}) > 0$.

Exercise 5.4. *Calculate the multifractal spectrum* ($\alpha \mapsto \dim \Lambda_\alpha$) *for the* $(p, 1-p)$ *Bernoulli measure supported on the Cantor middle-third set.*

Exercise 5.5. *Let* $f_1(x) = 1/2x$, $f_2(x) = 1/4x + 3/4$. *Let* $p_1 = p$ *and* $p_2 = 1-p$. *Determine an explicit formula for* $\beta(q)$ *and the multifractal spectrum (in terms of* q*).*

6 Projections of sets

The study of projections of sets and measures has a long history, with important results from Besicovitch in the mid-1930s. With the advent of dimension theory, studies did not consider just 1-sets, or k-sets for $k \in \mathbb{N}$ but also sets of non-integral dimension. The most famous result is *Marstrand's projection theorem (1956)* which gives an almost sure result on the Hausdorff dimension of projections of sets in the plane. The original proof was a very intricate geometric argument, which was later proved with a much shorter potential theoretic approach by Kaufmann. Mattila later extended this result to higher dimensions. In this section we will only give a proof in the plane for the first part of the theorem, concerning the dimension. The absolute continuity part of the theorem is outside of the scope of this course as it requires the study of Fourier transforms of measures.

We first motivate the study by recalling results from earlier sections. In Section 3 we established that the Hausdorff, packing, and box-counting dimensions are Lipschitz stable. That is, they do not increase under Lipschitz maps. Since orthogonal projections are Lipschitz (in fact they are 1 Lipschitz), we know that $\dim \pi F \leq \dim F$, where dim is any of those dimensions and π is an orthogonal projection. If $\pi \in G(n, m)$, the Grasmannian of orthogonal projections[11] from $\mathbb{R}^n \to \mathbb{R}^m$ we can improve this to $\dim \pi F \leq \min\{m, \dim F\}$ for all $F \in \mathbb{R}^n$ and $\pi \in G(n, m)$.

Clearly such an equality cannot hold in general. The line $L = \{(t, t+1) \in \mathbb{R}^2 : t \in \mathbb{R}\}$ has dimension 1 and projections π_θ onto the line at angle θ to the x-axis is $\mathbb{R} = \pi_\theta L$ for all $\theta \in [0, \pi)$ with the exception of $\theta = \arctan(1) = \pi/4$. The same applies to more complex sets such as the self-similar set invariant under $x \mapsto x/3, x \mapsto x/3 + (2/3, 0), x \mapsto x/3 + (1/3, 2/3)$ which has Hausdorff dimension 1 and is totally disconnected. Its projection onto the x-axis is however $[0, 1]$ and the Cantor middle third set onto the y-axis. It turns out to be true, however, that the simple upper bound for the dimension is sharp for most projections. This result is known as Marstrand's projection theorem.

Theorem 6.1 (Marstrand Projection Theorem)**.** *Let* $F \subset \mathbb{R}^2$ *with Hausdorff dimension* $s = \dim_H F$. *Then, for almost all* $\theta \in [0, \pi)$,

$$\dim_H \pi_\theta F = \min\{1, s\}.$$

Further, if $s > 1$*, the set* $\pi_\theta F$ *has positive* 1*-dimensional Lebesgue measure for almost all* θ*.*

[11] Many authors exchange the role of the two spaces in their terminology and write $G(m, n)$ for all orthogonal projections from $\mathbb{R}^n$ into $\mathbb{R}^m$.

Theorem 6.2 (Higher dimensional Projection Theorem). *Let $F \subset \mathbb{R}^d$ with Hausdorff dimension $s = \dim_H$. Fix $n \in [1, d-1]$, then*

$$\dim_H \pi F = \min\{n, s\}$$

for ν-almost all $\pi \in G(d,n)$, where ν is the natural volume measure on $G(d,n)$.

Further, if $s > n$, then $\mathcal{H}^n(\pi F) > 0$ for ν-almost all $\pi \in G(d,n)$.

We will prove the first part of Marstand's Projection Theorem.

Proof of Marstrand's Projection Theorem. Recall that the t-energy of a measure μ is given by

$$I_t(\mu) = \iint \frac{d\mu(x)d\mu(y)}{|x-y|^t}.$$

Frostman's lemma for energies states that the existence of a finite measure μ on a set E with finite t-energy implies that the Hausdorff dimension of E is at least t. Further any Borel set with positive t-Hausdorff measure supports a finite measure μ on E with finite t-energy.

Let $t < \min\{1, s\}$. Then $\mathcal{H}^t(F) = \infty$ and so there exists a finite measure μ on F such that $I_t(\mu) < \infty$. Consider the measure μ_θ, the pushforward of μ under π_θ. In particular, for every measurable E, $\mu_\theta(E) = \mu(\{x \in F : \pi x \in E\}$. Equivalently,

$$\int_{-\infty}^{\infty} f(x)d\mu_\theta(x) = \int f(\pi_\theta x)d\mu(x) = \int f(x \cdot \vec{\theta})d\mu(x),$$

where $\vec{\theta}$ is the unit vector in direction θ. We now consider the energy $I_t(\mu_\theta)$ of this projected measure that is supported on $\pi_\theta F$. Integrating the energy with respect to the angle gives

$$\begin{aligned}
\int_0^\pi I_t(\mu_\theta)d\theta &= \int_0^\pi \iint_{\mathbb{R}\times\mathbb{R}} \frac{d\mu_\pi(u)d\mu_\pi(v)}{|u-v|^t}d\theta \\
&= \int_0^\pi \iint_{F\times F} \frac{d\mu(x)d\mu(y)}{|x\cdot\vec{\theta} - y\cdot\vec{\theta}|^t}d\theta \\
&= \iint_{F\times F} \int_0^\pi \frac{d\theta}{|\cos(\phi-\theta)|^t}\frac{d\mu(x)d\mu(y)}{|x-y|^t},
\end{aligned}$$

where ϕ is the angle between the x-axis and $x - y$. The integral $\int_0^\pi |\cos(\phi-\theta)|^{-t}d\theta$ does not depend on θ and since $\cos(t) \sim t$ near $\pi/2$, and $t < 1$, the integral is bounded and there exists some c_t only depending on t such that

$$\int_0^\pi I_t(\mu_\theta)d\theta = c_t \iint_{F\times F} \frac{d\mu(x)d\mu(y)}{|x-y|^t} = c_t I_t(\mu) < \infty.$$

This shows that $I_t(\mu_\theta) < \infty$ for almost every $\theta \in [0, \pi)$, proving the first part of the theorem. □

6.1 Other dimensions

While the upper bound holds for the box-counting dimension, there is no such nice theorem for the box-counting dimension. One can state results in terms of "dimension profiles" which are a generalisation using potential theory of the box-counting notion. For more info, see [6].

For the lower dimension and the Assouad dimension, there is also no analogue. For the Assouad dimension, we still require an example of a set that is not Lipschitz stable. Let $f_1(x) = 1/3x, f_2(x) = 1/3x + (2/3, 0), f_3(x) = 1/4x + (0, 1/2)$. This iterated function system in the plane satisfies the strong separation condition and so its Hausdorff and Assouad dimension is the similarity dimension s which satisfies $2/3^s + 1/4^s = 1$. This gives $s \approx 0.92611 \cdots < 1$. However, projecting this set on the x-axis gives the set invariant under the projected IFS $x \mapsto x/3, x \mapsto x/3 + 2/3, x \mapsto x/4$ which does not satisfy the weak separation condition (since $\log 4/\log 3 \notin \mathbb{Q}$) and so has Assouad dimension 1. But then $\dim_A \pi_0 F = 1 > \dim_A F$ and the Assouad dimension is not Lipschitz stable.

6.2 An application to self-similar sets

Previously we claimed that the similarity dimension is the "best guess" to the Hausdorff dimension of self-similar sets. We proved that this is true when the OSC applies and that it seems likely we can only get a drop when we have exact overlaps (dimension drop conjecture). There is another reason why we may consider the dimension the appropriate notion. If we were to take a self-similar set at random, the self-similar set we obtain does have Hausdorff dimension equal to the similarity dimension of the set, almost surely. This is made precise in this theorem

Theorem 6.3. *Let $\{L_i\}_{i=1}^N$ be a collection of linear maps of the form $L_i : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto \lambda_i x$, where $\lambda_i < 1/3$. Let s be the associated similarity dimension satisfying $\sum_i \lambda_i^s = 1$ and let $\mathbf{t} = (t_1, \ldots, t_N) \in (\mathbb{R}^d)^N$ be a vector of translation vectors for each $i \in \{1, \ldots, N\}$. Let $F_{\mathbf{t}}$ be the self-similar set invariant under the IFS $\{L_i(x) + t_i\}$. Then, $\dim_H F_{\mathbf{t}} = \min\{s, d\}$ for $\mathcal{L}^{d\cdot N}$-almost every $\mathbf{t} \in \mathbb{R}^{d\cdot N}$ and if $s > d$, we have $\mathcal{L}^d(F_{\mathbf{t}}) > 0$ for $\mathcal{L}^{dN}$-almost every $\mathbf{t}$.*

Proof. For simplicity we only prove the $d = 1$ $N = 2$ case fully and indicate how the entire theorem follows analogously (though requires much more notation!). We can reduce the case to $t_i \in [-C, C]$ for $C \geq 1$ large enough since if it was not Lebesgue almost sure, there must be a C for which the result does not holds almost surely. Let $\tau_1 = (1, 0)$, $\tau_2 = (0, 1)$ be the standard orthogonal basis of $\mathbb{R}^{d\times N} = \mathbb{R}^2$. Now consider the iterated function system given by the two functions $f_1(x) = L_1(x) + C\tau_1$ and $f_2(x) = L_2(x) + C\tau_2$. Since the maps contract by a factor greater than 3, we can find R such that $f_i(B(0, R)) \subset B(0, R)$ and $f_i(B(0, R)) \cap f_j(B(0, R)) = \varnothing$ since the former requires $C + c_i R < R \Rightarrow R > C/(c_i - 1) > (3/2)C$ and the latter that $c_i R < C/2 \Rightarrow R < C/(2c_i) > (3/2)C$. Hence, choosing $R \in ((3/2)C, \min_i C/(2c_i))$ is sufficient (and possible). Thus we see that this higher dimensional IFS satisfies the SSC and has attractor with Hausdorff dimension equal to the similarity dimension.

Observe now that we can project the set and iterated function system by π_θ to get the projected IFS $\pi_\theta f_1(x) = L_1(x) + C \cos\theta, \pi_\theta f_2(x) = L_2(x) + C \sin\theta$. Hence, by varying $\theta \in [0, \pi)$ and translating, we obtain every every iterated function system $f_i(x) + t_i$ with $t_i \in [-C, C]$. Applying the Marstrand projection theorem to our higher dimensional IFS, we see that its projection has dimension $\min\{1, s\}$, where s is the similarity dimension. But then, almost every IFS of the form $f_i(x) + t_i$ must have the required dimension for almost every t_i as cos is absolutely continuous w.r.t. Lebesgue.

The higher dimensional analogue uses translations in all $d \times N$ dimensions and then projects into $\mathbb{R}^d$ instead of $\mathbb{R}^1$, giving the required result. $\square$

6.3 Digital sundial

Instead of asking what projections of arbitrary sets are, we could also ask whether it is possible to construct a set with specified projections. In fact, this is possible, as long as we are happy with projections being "almost surely" what we want.

Theorem 6.4. *For all $\theta \in [0, \pi)$, let P_θ be a subset of $\mathbb{R}$ such that the set $\bigcup_\theta \{(\theta, y) : y \in P_\theta\}$ is $\mathcal{L}^2$-measurable. Then there exists a Borel set $F \subseteq \mathbb{R}^2$ satisfying $P_\theta \subseteq \pi_\theta F$ for all θ and $\mathcal{L}^1(\pi_\theta \setminus P_\theta) = 0$ for almost all θ.*

Heuristics of Proof. Since we are only concerned with the Lebesgue measure of the projection, we can restrict our attention to sets whose projections are intervals. Let $A = [\phi, \phi + \delta)$ be a range of projection angles. We can construct a set E such that $\pi_\theta E$ is a line segment for $\theta \in A$, whereas $\pi_\theta E$ is negligible for all other θ. Let E_0 be a line of length λ at angle ϕ. Choose $k \in \mathbb{N}$ large and $\varepsilon > 0$ small. We can subdivide E_0 into k many intervals of length approximately λ/k at an angle of ε to E_0. We call this collection of k lines E_1. We subdivide these k intervals further, replacing them with k intervals of length approximately λ/k^2 at an angle of ε to the angle of the E_1 intervals to get E_2. We continue $K = \lceil \delta/(2\varepsilon) \rceil$ times to get a set k^K intervals at an angle of approximately $\theta + \delta/2$. Comparing the projections of E_K with that of E_0 we see that for angles $\theta \in A$ the projections coincide, whereas for all other directions the projections are small.

This idea can be expanded upon to give sets with projections close to P_θ in arbitraily narrow bands of directions and taking unions of such sets we get approximations for all our required projections. Taking limits in construction depths gives sequences of compact sets which have convergent subsequences with our required properties. □

This strategy can be employed in higher dimensions giving us the (theoretical) possibility to build a digital sundial: Let (θ, ϕ) be all possible angles of the sun. We can, for narrow enough ranges of these angles, prescribe that $\pi_{\theta,\phi}$ corresponds to the $\mathcal{L}^2$ positive set of a digital readout of time and date when projected onto the 2-dimensional "ground. The "sundial theorem" says that it is possible to create a compact set $F \subset \mathbb{R}^d$ that provides a digital read out of the current date and time.

6.4 More recent results

More recently focus has shifted on Marstrand type projection results in three main ways:

1. Projections of other dimensions.
2. Size of the exceptional set.
3. Sets for which Marstrand's projection theorem holds for all angles.

In particular, it was shown that many sets (e.g. self-similar sets under mild assumptions) have projections that agree with Marstrand's result up to a set of exceptional angles of Hausdorff dimension 0. Assuming some "irrationality" of the projections, this can be improved further to all angles.

Theorem 6.5. *Let $\{f_i\}$ be a self-similar IFS of the form $f_i(x) = c_i O_i x + t_i$, where $O \in O(d)$. Let s be the similarity dimension of the associated attractor F. If the group generated by the individual orthogonal components $\langle O_i : 1 \le i \le N \rangle$ is dense in $O(d)$ (or even in $SO(d)$) then $\pi F = \min\{n, s\}$, for all $\pi \in G(d, n)$.*

Contrary, if $\langle O_i \rangle$ is finite and $s < d$, there exists at least one direction π_0 such that $\dim_H \pi_0 F < s$.

While one can show that there is no such result for the Assouad dimension itself (there are examples for which the Assouad dimension satisfies the Marstrand projection theorem for positive Lebesgue measure set of projections which is not full) there are some examples where it does. For instance, the projection of Mandelbrot percolation is full for all projections simultaneously, for almost every Mandelbrot percolation structure.

We shall not prove these more recent results in this course.

6.5 Exercises

Exercise 6.1. *Let $F \subset [0,1]$ be a self-similar set in the line. Consider the set $E = \{e^{2\pi i x} : x \in F\}$. Find $\dim_H \pi_\theta E$ for all $\theta \in [0, \pi)$.*

Exercise 6.2. *Show that $\dim_H \pi_\theta F \geq \dim_H F - 1$ for all $F \subseteq \mathbb{R}^2$ and all $\theta \in [0, \pi)$.*

Exercise 6.3. *Let $E, F \subseteq \mathbb{R}$. Consider the set $E + \lambda F = \{x + \lambda y : x \in E, y \in F\}$ and show that for almost every $\lambda \in \mathbb{R}$ we have $\dim_H E + \lambda F = \min\{1, \dim_H(E \times F)\}$.*

7 Bounded distortion and pressure

Recall that we say that an iterated function system is self-conformal if it constitutes only conformal contracting diffeomorphisms of $\mathbb{R}^d$ such that its derivative is Hölder continuous. As a matter of fact, the Hölder continuity is implied in dimensions $d \geq 2$ from the conformality, whereas "conformality" is defined as having Hölder continuous derivative for contractions in $\mathbb{R}$.

7.1 Bounded distortion

This Hölder continuity leads to the principle of bounded distortion, stating that any image f_v does not distort by more than a global constant. Throughout this section we write $c_{\max} = \sup_i \sup_x |f_i'(x)|$ and $c_{\inf} = \inf_i \inf_x |f_i'(x)|$.

Lemma 7.1 (Principle of bounded distortion)**.** *Let $\{f_i\}$ be a self-conformal IFS. Then there exists a constant $D > 0$ such that f_v satisfies*

$$(1/D)\|f_v'\|_\infty \leq \|f_v'(x)\| \leq \|f_v'\|_\infty$$

for all $v \in \Sigma_*$ and x in the domain. In particular this implies that the derivative is uniformly comparable for all points in the domain. This then implies

$$(1/C)\|f_v'\|_\infty |x - y| \leq |f_v(x) - f_v(y)| \leq \|f_v'\|_\infty |x - y|$$

for all $v \in \Sigma_*$ and some universal constant C.

Proof. The upper bound is immediate from the definition. Let x be an arbitrary point in the domain and let y be such that $|f_v'(y)| = \|f_v'\|_\infty$ as we may assume the domain is compact. Then, for $v \in \Sigma_n$ and using the chain rule repeatedly,

$$\begin{aligned} |f_v'(y)| &= \left| f_{v_1 \dots v_{n-1}}'(f_{v_n}(y)) \cdot f_{v_n}(y) \right| \\ &= \left| \prod_{i=1}^{n} f_{v_i}'(f_{v_{i+1} \dots v_n}(y)) \right| \end{aligned}$$

$$= \left| \prod_{i=1}^{n} f'_{v_i}(f_{v_{i+1}\dots v_n}(x) + \delta_i) \right|$$

for some $|\delta_i| \leq c_{\max}^{n-i} \operatorname{diam}(F)$ since $|f_{v_{i+1}\dots v_n}(x) - f_{v_{i+1}\dots v_n}(y)| \leq c_{\max}^{n-i} \operatorname{diam}(F)$. Using α-Hölder continuity,

$$= \left| \prod_{i=1}^{n} (f'_{v_i}(f_{v_{i+1}\dots v_n}(x)) + \Delta_i) \right|$$

for some $|\Delta_i| \leq C|\delta_i|^\alpha$. This gives

$$\begin{aligned}
&= \left| \prod_{i=1}^{n} (f'_{v_i}(f_{v_{i+1}\dots v_n}(x)) \cdot \prod_{i=1}^{n} \left(1 + \frac{\Delta_i}{f'_{v_i}(f_{v_{i+1}\dots v_n}(x))} \right) \right| \\
&\leq |f'_v(x)| \cdot \left| \prod_{i=1}^{n} \left(1 + \frac{C|c_{\max}^{n-i} \operatorname{diam}(F)|^\alpha}{c_{\min}} \right) \right| \\
&\leq |f'_v(x)| \exp\left(\sum_{i=1}^{n} \frac{C \operatorname{diam}(F)^\alpha}{c_{\min}} c_{\max}^{\alpha i} \right) \leq D|f'_v(x)|
\end{aligned}$$

for some $D > 0$ as the sum is uniformly bounded. □

We immediately get

Corollary 7.2. *Let $\{f_i\}$ be self-conformal. Then,* $\operatorname{diam} f_v(F) \sim \|f'_v\|_\infty$.

7.2 Pressure

Using this relationship between the derivative and diameters of cylinders, we can make a guess as to the Hausdorff dimension of the associated attractor. We can cover F with $f_v(F)$, where $v \in \Sigma_n$ for each n. Since the derivatives and diameters are related and $|f'_v| \leq c_{\max}^n$, we have

$$\mathcal{H}^s_{c_{\max}^n}(F) \leq \sum_{v \in \Sigma_n} \operatorname{diam}(f_v(F))^s \lesssim \sum_{v \in \Sigma_n} \|f'_v\|_\infty^s. \tag{7.1}$$

This sum diverges to infinity for small s at an exponential rate and converges to 0 for large s, again exponentially in n. There exists a critical value s_0 where this behaviour changes. We refer to the exponential rate of the sum as the pressure of the IFS.

Definition 7.3. *Let $\{f_i\}$ be a conformal IFS. The* ***pressure*** *of the IFS is given by*

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{v \in \Sigma_n} \|f'_v\|_\infty^s$$

We will show that this is well-defined in a bit, but first we comment that the sum in (7.1) converges to 0 for all s such that $P(s) < 0$. This shows that $\dim_H F \leq s_0$, where $s_0 = \inf\{s : P(s) < 0\}$. In fact, s_0 is the unique value for which $P(s_0) = 0$.

Proposition 7.4. *The limit in the definition of the pressure exists. The function $P(s)$ is continuous for $s \in [0, \infty)$, decreasing and has a unique zero.*

Exercise 7.1. *Prove these properties.*

Exercise 7.2. *Prove that a self-conformal set is quasi self-similar.*

7.3 The Hausdorff dimension of self-conformal sets

Perhaps unsurprisingly, the Hausdorff dimension is given by the zero of the pressure when there is sufficient separation between cylinders.

Theorem 7.5. *Let $\{f_i\}$ be a self-conformal iterated function system. Let $P(s)$ and F be the associated pressure function and attractor. Let s_0 be the unique value for which $P(s_0) = 0$. Then,* $\dim_H F = \dim_B F \leq s_0$ *and if F additionally satisfies the open set condition,* $\dim_H F = \dim_B F = s_0$.

Proof. Since any self-conformal set is quasi self-similar, the Hausdorff and box-counting dimensions coincide. The upper bound was shown in the text above (a simple covering argument).

The lower bound needs the construction of an appropriate measure, that gives each cylinder $f_v(F)$ mass comparable to $\|f_v'\|_\infty$. This measure can be constructed as the limit of discrete measures μ_n by setting

$$\mu_n(f_v(F)) = \frac{1}{S_n} \sum_{vw\in\Sigma_n} \|f_{vw}'\|_\infty^s,$$

where

$$S_n = \sum_{u\in\Sigma_n} \|f_u'\|_\infty^s.$$

Using the fact that any sequence of probability measures on a compact set $F \subset \mathbb{R}^d$ has a weakly convergent subsequence (Prokhorov's theorem), we set μ to be that measure.

Using the chain rule,

$$f_{vw}'(x) = f_v'(f_w(x))f_w'(x)$$

and so $\|f_{vw}'\|_\infty^s \leq \|f_v'\|_\infty^s \|f_w'\|_\infty^s$. Further, $|f_v'(x)| \geq \|f_v'\|_\infty^s/D$ and so

$$D^{-2}\|f_v'\|_\infty^s \|f_w'\|_\infty^s \leq \|f_{vw}'\|_\infty^s \leq \|f_v'\|_\infty^s \|f_w'\|_\infty^s.$$

Summing over all words, we obtain $D^{-2}S_nS_m \leq S_{n+m} \leq S_nS_m$ and so for $v \in \Sigma_k$,

$$\mu_{n+k}(f_v(F)) = \frac{1}{S_{n+k}} \sum_{vw\in\Sigma_{n+k}} \|f_{vw}'\|_\infty^s \leq \frac{D^2}{S_nS_k}\|f_v'\|_\infty^s \sum_{w\in\Sigma_n} \|f_w'\|_\infty^s = D^2\frac{\|f_v'\|_\infty^s}{S_k}.$$

Similarly,

$$\mu_{n+k}(f_v(F)) \geq \frac{1}{S_nS_k} D^{-2}\|f_v'\|_\infty^s \sum_{w\in\Sigma_n} \|f_w'\|_\infty^s = D^{-2}\frac{\|f_v'\|_\infty^s}{S_k}.$$

Since this holds for all $n \in \mathbb{N}$, the weak limit μ satisfies

$$\frac{D^{-2}}{S_k} \leq \frac{\mu(f_v(F))}{\|f_v'\|_\infty^s} \leq \frac{D^2}{S_k}. \tag{7.2}$$

Finally, we relate S_k back to the pressure. Recall Fekete's lemma: for any subadditive sequence (a_k) the limit $\lim_k a_k/k$ exists and equals $\inf_k a_k/k$ (which may be $-\infty$). Hence $P(s) = \lim_n (1/n)\log S_n = \inf_n (1/n) \log S_n$ and so $S_n \geq \exp nP(s)$. Similarly, we may apply Fekete's lemma to $a_n = \log(D^2/S_n)$ as

$$a_{n+m} = 2\log D - \log S_{n+m} \leq 2\log D - \log D^{-2}S_nS_m$$

$$= 2\log D - \log S_n + 2\log D - \log S_m = a_n + a_m.$$

Thus,

$$a_n/n = (1/n)(\log D^2/S_n) \to -P(s) = \inf_n((2/n)\log D - (1/n)\log S_n).$$

We obtain $S_n \leq D^2 \exp nP(s)$. We have shown that S_n is comparable to $nP(s)$. Combining this with (7.2) gives

$$D^{-4} \leq \frac{\mu(f_v(F))}{\|f_v'\|_\infty \exp(-nP(s))} \leq D^2.$$

Thus, for s_0 such that $P(s_0) = 0$ we get a measure that satisfies our assumption. The rest of the proof is a standard mass distribution argument, combined with a volume lemma. □

7.4 A primer to thermodynamic formalism

You may wonder why we have expressed the measure in terms of the pressure, rather than using $P(s_0) = 0$ directly. The pressure above can be generalised to different functions (known as potentials) that measure different aspects of the dynamics on the set.

In more generality, we replace the $\|f_v'\|_\infty$ term by $\exp\sum_{k=0}^{n-1}\phi(f_{\sigma^k v}x_v)$, where σ is the one-sided shift and ϕ is a general potential function and x_v is the fixed point of $f_v(x_v) = x_v$. For $\phi(x) = s\log|f_i'(y)|$, where $f_i(y) = x$, we obtain the same expression as before, as $\exp\sum_{k=0}^{n-1}\phi(f_{\sigma^k v}x_v) = \exp\log|f_v'(x_v)| \sim \|f_v'\|_\infty$ by the chain rule.

Assuming that ϕ is a Hölder function on F, we obtain a similar bounded distortion principle and we can define a measure μ in terms of the potential ϕ.

Proposition 7.6 (Principle of bounded variation)**.** *Let $\phi : F \to \mathbb{R}$ be a Hölder function and let $v \in \Sigma_n$. Then there exists $B > 0$ such that for all $k \leq n$,*

$$\left|\sum_{j=1}^{k}\phi(f_{\sigma^j v}x) - \sum_{j=1}^{k}\phi(f_{\sigma^j v}y)\right| \leq B\,\mathrm{diam}(F)^{-1}\,\mathrm{diam}(f_{v_{k+1}\dots v_n}(F))$$

for all $x, y \in f_v(F)$.

This gives rise to a notion of pressure, called the topological pressure, and an associated measure, called the Gibbs measure.

Proposition 7.7. *Let $\phi : F \to \mathbb{R}$ be a Hölder potential. Then the limit*

$$P(\phi) = \lim_n \frac{1}{n}\log\sum_{v\in\Sigma_n}\exp\sum_{k=1}^{n}\phi(f_{\sigma^k v}x_v)$$

*exists and does not depend on $x_v \in f_v(F)$ (by convention we take x_v to be the unique fixed point of f_v). We call $P(\phi)$ the **topological pressure** of the potential ϕ.*

*Further, there exists a Borel probability measure μ called the **Gibbs measure** of the potential ϕ such that*

$$A^{-1} \leq \frac{\mu(f_v(F))}{\exp(-nP(\phi) + \sum_{k=1}^{n}\phi(f_{\sigma^k v}x_v))} \leq A$$

for some uniform A.

Choosing appropriate potentials gives information about the attractor. Its size, for instance, can be gleamed by setting ϕ to be the derivative (known as the geometric potential), whereas the multifractal spectrum of the Gibbs measure associated with potential ψ can be found by defining $\beta(q)$ implicitly by $P(\beta(q)\phi + q\psi) = 0$, where ϕ is the geometric potential. The multifractal spectrum is then given by $\beta'(q) \mapsto -\beta'(q)q + \beta(q)$, analogous to the self-similar formula.

In fact, this shows that self-conformal sets look like self-similar sets "in the limit". Much that holds for self-similar sets also applies to self-conformal sets. In particular, if we define the WSC by considering maps restricted to the attractor (instead of the whole domain or the unit cube) Theorem 4.27 and Corollary 4.28 also apply to self-conformal sets, where the pressure is taken over non-superfluous words. You may want to try Exercise 4.8 in the conformal setting which is significantly harder to prove for conformal maps and relies on much more delicate estimates. For full proofs in the conformal setting see [1].

Exercise 7.3. *(optional) Prove both propositions.*

7.5 Where it all fails: self-affine sets (optional)

Recall that a set is called self-affine if it is invariant under an IFS $\{f_i\}$, where f_i is of the form $f_i(x) = A_i x + t_i$, where A_i is a non-singular matrix with norm less than 1. We can look at a simple example that satisfies the strong separation condition/open set condition.

Fix $2 \le m < n$ and let $A_i = \begin{pmatrix} 1/n & 0 \\ 0 & 1/m \end{pmatrix}$. Let $t_{i,j} = \begin{pmatrix} i/n \\ j/m \end{pmatrix}$ be a translation vector. The iterated function system $\{Ax + t_{i,j}\}_{(i,j)\in D}$ for some digit set $D \subset \{1, \dots, n\} \times \{1, \dots, m\}$ is knows as a self-affine iterated function system of Bedford-McMullen type and the associated attractor is referred to as a Bedford-McMullen set (or carpet) for short.

The digit set D can be chosen such that the iterated function system is strictly self-affine and is our first example for an invariant set, where Hausdorff, box-counting, and Assouad dimension all differ. To give the dimension formula more concisely we need to introduce some more notation. Let $R_k = \#\{(i,j) \in D : j = k\}$ be the number of maps into row k. Let R be the number of non-empty rows $R = \#\{k : R_k > 0\}$, the dimensions are:

Theorem 7.8. *Let F be a Bedford-McMullen carpet as above. The Hausdorff, box-counting, and Assouad dimensions are*

$$\dim_H F = \frac{\log \sum_j R_j^{\log m/\log n}}{\log m}, \quad \dim_B F = \frac{\log R}{\log m} + \frac{\log(\#D/R)}{\log n},$$

and

$$\dim_A F = \frac{\log R}{\log m} + \max_k \frac{\log R_k}{\log n}.$$

[Proof idea here]

8 Bonus: A fractal proof of the infinitude of primes

We now give a brief proof of the infinitude of primes, inspired by dimension theory. It hinges mostly on the following theorem that you can easily prove yourself.

Proposition 8.1. *Let $A, B \subset \mathbb{R}$ be bounded sets. Then, $\overline{\dim}_B A \cdot B \le \overline{\dim}_B A + \overline{\dim}_B B$, where $A \cdot B = \{a \cdot b : a \in A, b \in B\}$.*

Theorem 8.2. *Let $\mathbb{P}$ be the set of all primes. Then $\#\mathbb{P} = \infty$.*

Proof. We first note that the set of powers of $1/n$, $P(n) = \{1/n^k : k \in \mathbb{N}_0\}$ has zero box-counting dimension. This follows as $P(n)$ can be covered by $[0, r] \cup \bigcup_{k=0}^{\lceil \log(r/2)/\log n \rceil} B(1/n^k, r/2)$. Hence, $N_r(P(n)) \le 2 + \log(r/2)/\log n$ and

$$\overline{\dim}_B P(n) \le \lim_{r \to 0} \frac{\log(2 + \log(r/2)/\log n)}{-\log r} = 0.$$

Note that $\mathbb{N} = \prod_{p \in \mathbb{P}} \{p^k : k \in \mathbb{N}_0\}$ and so $1/\mathbb{N} = \prod_{p \in \mathbb{P}} P(p)$. Now assume $\mathbb{P}$ is finite, then using the Proposition inductively,

$$\overline{\dim}_B 1/\mathbb{N} = \overline{\dim}_B \prod_{p \in \mathbb{P}} P(p) \le \sum_{p \in \mathbb{P}} \overline{\dim}_B P(p) = 0.$$

But $\dim_B 1/\mathbb{N} = 1/2$, a contradiction and our claim follows. □

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