Geometric Measure Theory and Dynamics

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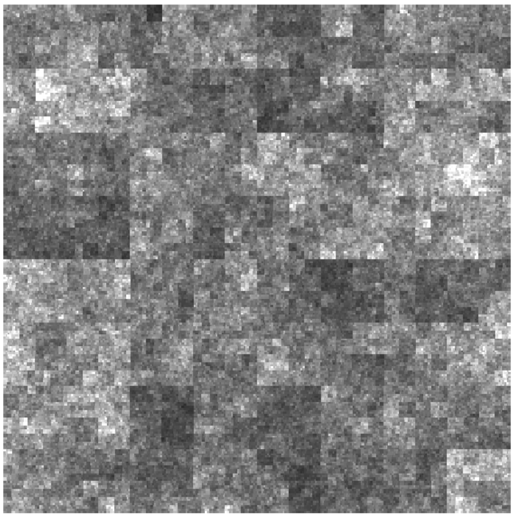

Abstract

Geometric measure theory explores the properties of general sets with only limited structure through measure theory. Measure theory and advanced techniques are needed to analyse the complicated structures that arise in contexts such as dynamical systems, which generally are not "smooth". While complex, many spaces (such as Ahlfors regular spaces and Kakeya sets) still retain much 'hidden structure' that we will explore in this course through the lens of measure theory.[1]

The title image shows the density of a random cascade measure. A useful random probability measure supported on the unit square.

[1]These lecture notes will evolve throughout term and there may be frequent significant structural changes. I have no doubt that there are many typos and inaccuracies in this manuscript. If you find anything that would need correction, please let me know at sascha.troscheit@univie.ac.at. Thank you!

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Aims and Motivation

During the course our main aim is to develop measure theoretic tools to analyse geometric object, especially invariant sets from dynamical systems. A central object of study is the **Hausdorff measure**, an outer measure that generalises the Lebesgue dimension to non-integer "dimensions", as well as arbitrary metric spaces. For the moment, we will omit its definition and collect some key properties when used as a measure on $\mathbb{R}^d$.

Let $F \subseteq \mathbb{R}^d$ and $s \geq 0$. The Hausdorff measure $\mathcal{H}^s$

- is translation invariant: $\mathcal{H}^s(F) = \mathcal{H}^s(F+t)$ for $t \in \mathbb{R}^d$;
- has appropriate scaling: $\mathcal{H}^s(c \cdot F) = c^s \cdot \mathcal{H}^s(F)$ for all $c > 0$;
- is equivalent to the Lebesgue measure $\mathcal{L}^s$, when $s \in \mathbb{N}$;
- is the counting measure when $s = 0$.

A motivation. Many invariant sets in dynamical systems are "very small" compared to the size of the underlying state space. We endeavour to better determine their structure.

Let (X, d) be a metric space, $T : X \to X$ be a self-map (a surjective mapping).

Definition 0.1. *The tuple (X, T) is called a (discrete) dynamical system with state space X and dynamic T.*

Definition 0.2. *We say that a set $F \subseteq X$ is (forward or positively) invariant under T if $T(F) = F$.*

Example. Let $X = [0,1]/_{0\sim 1}$ be the circle and let

$$T(x) = 3x \mod 1 = 3x - \lfloor 3x \rfloor$$

be the tripling map. Clearly, the whole space X is invariant as $T : X \to X$ is surjective. On the other hand $[0, \frac{1}{3}]$ is not invariant as $T[0, \frac{1}{3}] = X$.

Now let

$$F = \left\{ x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} : a_i \in \{0,2\}\, \forall i \in \mathbb{N} \right\} \setminus \{1\}$$

be the collection of all points in $[0,1)$ with ternary expansions only containing the digits 0 and 2. We now show that F is also invariant under T. Note that

$$\begin{aligned}
T(F) &= \left\{ T(x) : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, a_i \in \{0,2\}\, \forall i \in \mathbb{N} \right\} \setminus \{1\} \\
&= \left\{ 3 \sum_{i=1}^{\infty} \frac{a_i}{3^i} \mod 1 : a_i \in \{0,2\}\, \forall i \in \mathbb{N} \right\} \setminus \{1\} \\
&= \left\{ a_1 + \sum_{i=1}^{\infty} \frac{a_{i+1}}{3^i} \mod 1 : a_i \in \{0,2\}\, \forall i \in \mathbb{N} \right\} \setminus \{1\} \\
&= \left\{ \sum_{i=1}^{\infty} \frac{a_{i+1}}{3^i} : a_i \in \{0,2\}\, \forall i \in \mathbb{N} \right\} \setminus \{1\} \\
&= F
\end{aligned}$$

and so F is invariant under T. It is not too hard to show that the Lebesgue measure is 0. (Exercise!) Can we gain more information using the Hausdorff measure?

Notice that $F' = F \cup \{1\}$ can be partitioned by the first digit in its ternary expansion. It can be written as the union

$$F' = (\tfrac{1}{3}F' + 0) \cup (\tfrac{1}{3}F' + \tfrac{2}{3}).$$

Since this union is disjoint we can use that $\mathcal{H}^s$ is a measure[2], that it is translation invariant, and use its scaling properties to obtain the following equality:

$$\mathcal{H}^s(F) = \mathcal{H}^s(\tfrac{1}{3}F + 0) + \mathcal{H}^s(\tfrac{1}{3}F + \tfrac{2}{3}) = 2 \cdot \tfrac{1}{3^s}\, \mathcal{H}^s(F).$$

This equality is trivially satisfied if $\mathcal{H}^s(F)$ is zero or infinite. However, assuming that there is an $s \geq 0$ such that $\mathcal{H}^s$ is positive and finite, we must have – upon division by $\mathcal{H}^s$ – that

$$1 = \frac{2}{3^s} \iff s = \frac{\log 2}{\log 3}.$$

Telling us that any non-trivial "natural" measure must be $\log_3 2$-dimensional.

Remark. As it turns out the $\log_3 2$-dimensional Hausdorff measure of F is positive and finite and is equivalent to an important measure of the dynamical system $T : F \to F$, the measure of maximal entropy.

Exercise 0.1. *Show directly from the definition of the Lebesgue measure that the invariant set F in the exercise above is* 0.

1 Basic Measure Theory: Definitions and some results.

Recall the definition of an outer measure and a measure.

Definition 1.1. *Let X be a set and write $\mathcal{P}(X) = \{A \subseteq X\}$ for its power set. An outer measure on X is a set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that*

1. $\mu^*(\varnothing) = 0$*;*
2. $\mu^*(A) \leq \mu^*(B)$ *for all* $A \subseteq B \subseteq X$*;*
3. $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ *for all countable collections* $A_1, A_2, \dots \in \mathcal{P}(X)$.

Definition 1.2. *Let X be a set and Σ be a σ-algebra over X (i.e. $\varnothing, X \in \Sigma$, and Σ is closed under taking complements and countable unions). A measure μ on the measure space (X, Σ) is a set function $\mu : \Sigma \to [0, \infty]$ that satisfies*

1. $\mu(\varnothing) = 0$*;*
2. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} A_i$ *for all countable, pairwise disjoint collections* $A_1, A_2, \dots \in \Sigma$.

[2] Technically, we also need to know that F' is measurable, but let's take this for granted now.

Remark. Relaxing condition (2) in the definition of measure to finite unions of pairwise disjoint collections gives rise to the notion of a *content*.

Following Mattila [1], we will not make a distinction between outer measures and measures and agree to call all outer measures simply a measure. This enables us to speak of the measure of any subset of X, albeit with the potential issue arising from measurability. In fact, any measure μ gives rise to an outer measure μ^* by defining

$$\mu^*(A) = \inf \{\mu(B) : A \subseteq B \in \Sigma\}.$$

Similarly, restricting μ^* to a σ-algebra of measurable subsets gives a measure μ.

Definition 1.3. *A set $A \subseteq X$ is said to be μ-measurable if*

$$\mu(B) = \mu(A \cap B) + \mu(B \setminus A) \quad \text{for all} \quad B \subseteq X.$$

We will now call outer measures simply measures and drop the superscript * from our notation.

1.1 Basic properties of measures and some definitions

Proposition 1.4. *Let μ be a measure on X and let $\mathcal{M} \subseteq \mathcal{P}(X)$ be the family of μ-measurable subsets of X.*

1. *$\mathcal{M}$ is a σ-algebra.*
2. *$\mu(A) = 0 \Rightarrow A \in \mathcal{M}$ for all $A \subset X$.*
3. *If $A_1, A_2, \dots \in \mathcal{M}$ are pairwise disjoint, then $\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i)$.*
4. *If $A_1, A_2, \dots \in \mathcal{M}$ then*

$$\lim_{i\to\infty} \mu(A_i) = \begin{cases} \mu\left(\bigcap_{i=1}^\infty A_i\right) & \text{if } A_1 \subseteq A_2 \subseteq \dots; \\ \mu\left(\bigcup_{i=1}^\infty A_i\right) & \text{if } A_1 \supseteq A_2 \supseteq \dots \text{ and } \mu(A_1) < \infty. \end{cases}$$

Proof. The proof is left as an exercise. □

Definition 1.5. *Let μ be a measure on (X, d).*

1. *μ is called regular if $\forall A \subseteq X, \exists B \in \mathcal{M}$ with $A \subseteq B$ and $\mu(A) = \mu(B)$.*
2. *μ is called Borel if all Borel sets are μ-measurable.*
3. *μ is called Borel regular if the sets B in (1) are Borel sets.*
4. *μ is called a Radon measure if it is Borel and*
   - *(a) $\mu(K) < \infty$ for all compact $K \subseteq X$.*
   - *(b) $\mu(U) = \sup \{\mu(K) : K \subseteq U, K$ is compact$\}$ for all open $U \subseteq X$.*
   - *(c) $\mu(A) = \inf \{\mu(U) : A \subseteq U, U$ open$\}$ for all $A \subseteq X$.*

Examples. The d-dimensional Lebesgue measure $\mathcal{L}^d$ on $\mathbb{R}^d$ is a Radon measure, as is the Dirac measure δ_x at $x \in X$. The counting measure $\mathcal{H}^0$ is Borel regular for all metric spaces (X, d) but Radon only if (X, d) is discrete, i.e. if all compact subsets of K have finite cardinality.

Theorem 1.6 (Carathéodory's criterion)**.** *Let μ be a measure on (X, d). Then μ is Borel if and only if*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for all $A, B \subseteq X$ with $d(A, B) := \inf \{d(a, b) : a \in A, b \in B\} > 0$.

Exercise 1.1. *Prove Proposition 1.4.*

Further we have the need to say when a measure converges. First, the portmanteau theorem, which gives many equivalent definitions for the convergence of a measure.

Theorem 1.7. *Let X be a metric space and Σ its Borel σ-algebra. A sequence of probability measures defined on (X, Σ) is said to converge weakly to a measure μ on (X, Σ) if any of the following equivalent statements hold:*

- $\int f(x) d\mu_n(x) \to \int f(x) d\mu$ *for all bounded, continuous functions $f : X \to \mathbb{R}$.*
- $\int f(x) d\mu_n(x) \to \int f(x) d\mu$ *for all bounded and Lipschitz functions $f : X \to \mathbb{R}$.*
- $\limsup_{n\to\infty} \mu_n(K) \leq \mu(K)$ *for all closed sets $K \subseteq X$.*
- $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$ *for all open sets $U \subseteq X$.*

The following result is a useful observation.

Theorem 1.8. *Let μ_n be discrete probability measures in $\mathbb{R}^d$. Then there exists a subsequence n_k such that $\mu_{n_k} \to \mu$, where μ is a Borel probability measure supported in $\mathbb{R}^d$.*

1.2 Other useful results

We collect some other useful results that we will later in the course.

Definition 1.9. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of reals. If $a_{n+m} \leq a_n + a_m$ for all $m, n \in \mathbb{N}$, we say that the sequence is* subadditive*. Similarly, if there exists a constant $c \in \mathbb{R}$ and the sequence satisfies $a_{n+m} \leq a_n + a_m + c$ we say the sequence is* quasi-subadditive *with constant c.*

Lemma 1.10 (Fekete's lemma)**.** *Let (a_n) be a subadditive sequence. Then, the limit*

$$a = \lim_{n\to\infty} \frac{a_n}{n}$$

exists, equals $\inf_n(a_n/n)$, and takes values in $[-\infty, \infty)$.

Proof. Fix some $k \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$ there exists unique $p \in \mathbb{N}_0$ and $q \in \{0, 1, \ldots, p-1\}$ such that $n = pk + q$. Using the subadditivity condition repeatedly, we obtain

$$a_n = a_{pk+q} \leq a_{pk} + a_q \leq a_{(p-1)k} + a_k + a_q \leq \cdots \leq pa_k + a_q$$

and so

$$\frac{a_n}{n} \leq \frac{pa_k + a_q}{pk + q} \leq \frac{a_k}{k} + \frac{a_q}{pk}$$

Taking $n \to \infty$ gives $p \to \infty$ and $\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_k}{k}$. Since k was arbitrary,

$$\inf_k \frac{a_k}{k} \leq \liminf_{k\to\infty} \frac{a_n}{n} \leq \limsup_{k\to\infty} \frac{a_n}{n} \leq \inf_k \frac{a_k}{k}$$

and the limit exists and equals the infimum. Hence we must also have $a = \lim_n a_n/n \in [-\infty, \infty)$ as $a_1/1$ is an upper bound to a and is finite. $\square$

Corollary 1.11. *Let (a_n) be a quasi-subadditive sequence with constant c. Then, $a = \lim_n a_n/n$ exists and $a_n \geq na - c$.*

Proof. Note that

$$a_{n+m} + b \leq (a_n + a_m + b) + b = (a_n + b) + (a_m + b)$$

and so the sequence $(a_n + b)$ is subadditive. Using Fekete's lemma we get

$$a = \lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_n + b}{n} = \inf_k \frac{a_k + b}{k}.$$

This proves the first part. The second part derives from $a = \inf_k (a_k + b)/k \leq (a_k + b)/k$ and so $a_k \geq ka - b$ as required. $\square$

2 The Hausdorff measure and content

To ease notation we write $|U|$ for the diameter of a set, i.e. $|U| = \sup\{d(x,y) : x, y \in U\}$. We use the convention that $|\varnothing| = 0$.

Definition 2.1. *Let (X, d) be a metric space and let $s \geq 0$ and $\delta > 0$. The s-dimensional Hausdorff δ-content is*

$$\mathcal{H}^s_\delta(X) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : X \subseteq \bigcup_{i=1}^{\infty} U_i, U_i \text{ open}, |U_i| < \delta \right\},$$

where the infimum is taken over all such countable open δ-covers of X.

Notice that for increasing $\delta > 0$ the infimum is taken over a larger family of covers. Since we are taken the infimum, this means that $\mathcal{H}^s_\delta$ is decreasing in δ. Taking limits, we arrive at the definition of the Hausdorff measure and content.

Definition 2.2. *Let (X, d) be a metric space. The s-dimensional Hausdorff content is*

$$\mathcal{H}^s_\infty(X) = \lim_{\delta\to\infty} \mathcal{H}^s_\delta(X) = \inf_{\delta>0} \mathcal{H}^s_\delta(X).$$

Equivalently,

$$\mathcal{H}^s_\infty(X) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : X \subseteq \bigcup_{i=1}^{\infty} U_i, U_i \text{ open} \right\}.$$

Definition 2.3. *Let (X, d) be a metric space. The s-dimensional Hausdorff measure is*

$$\mathcal{H}^s(X) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(X) = \sup_{\delta>0} \mathcal{H}^s_\delta(X).$$

We can summarise the relations between those quantities by

$$\mathcal{H}^s_\infty(X) \leq \mathcal{H}^s_\delta(X) \leq \mathcal{H}^s_{\delta'}(X) \leq \mathcal{H}^s(X)$$

for all $0 < \delta' < \delta < \infty$.

Proposition 2.4. *The Hausdorff measure is a Borel measure.*

Proof. The proof that $\mathcal{H}^s$ is a measure is left as an exercise.

We will use Carathéodory's criterion to show that the Hausdorff measure is Borel. Let $A, B \subset X$ such that $\rho = d(A, B) > 0$. By subadditivity of measures, we only need to show that $\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B)$.

First, for $0 < \delta < \rho/2$ and any δ-cover U_i of $A \cup B$, we claim that each covering set U_i cannot intersect both A and B. To prove this fact we assume without loss of generality that $U_i \cap A \neq \varnothing$. Then there exists $a \in A \cap U_i$ and for any $x \in U_i$ we have $d(a, x) < \delta$ since $|U_i| < \delta$. By assumption any $b \in B$ satisfies $d(a, b) \geq \rho$. Using the triangle inequality, $d(b, x) \geq d(a, b) - d(a, x) \geq \rho - \delta \geq \rho/2 > 0$ from which the claim follows.

We may also assume that $\mathcal{H}^s(A \cup B)$ is finite as otherwise there is nothing to prove. Let $\varepsilon > 0$ and let $\{U_i\}$ be a δ-cover of $A \cup B$ such that

$$\mathcal{H}^s(A \cup B) \geq \sum_{i=1}^{\infty} |U_i|^s - \varepsilon = \sum_{\substack{i \in \mathbb{N} \\ U_i \cap A \neq \varnothing}} |U_i|^s + \sum_{\substack{i \in \mathbb{N} \\ U_i \cap B \neq \varnothing}} |U_i|^s - \varepsilon \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) - \varepsilon.$$

Letting $\delta \to 0$ gives

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B) - \varepsilon.$$

However, since $\varepsilon > 0$ was arbitrary, we get the required result. □

Remark The Hausdorff content is unfortunately misnamed as such and is – in fact – not a "proper" content. However, it is a measure albeit with the caveat that the family of measurable sets is fairly small.

An illustrative example is the set $F = \{(x, y) : x \in [0, 1], y \in \{0, 1\}\} = L \cup (L + (0, 1))$ which are two copies of the unit line, translated. One can show that $\mathcal{H}^1_\infty(L + t) = 1$, independent off the translation $t \in \mathbb{R}^2$, whereas F can be covered by an (open) square with sidelength $1 + \varepsilon$ for any ε. This gives

$$\mathcal{H}^1_\infty(F) \leq \sqrt{2} < 2 = \mathcal{H}^1_\infty(L) + \mathcal{H}^1_\infty(L + (0, 1)).$$

We collect some more properties of the Hausdorff measure.

Proposition 2.5. *Let (X, d) be a metric space. The Hausdorff measure satisfies the following:*

1. *If $\mathcal{H}^s(X) = 0$ for some $s \geq 0$, then $\mathcal{H}^t(X) = 0$ for all $t > s$.*
2. *If $\mathcal{H}^s(X) = \infty$ for some $s > 0$, then $\mathcal{H}^t(X) = \infty$ for all $0 \leq t < s$.*
3. *There exists at most one $s \in (0, \infty)$ such that $\mathcal{H}^s(X) \in (0, \infty)$.*

Proof. The proof of this theorem follows from monotonicity type property.

We show that $\mathcal{H}^s(X) < \infty$ implies $\mathcal{H}^t(X) = 0$ for all $t > s$. Let $\varepsilon, \delta > 0$ and let $\{U_i\}$ be a countable open δ-cover of X such that

$$\mathcal{H}^s(X) \leq \sum_{i \in \mathbb{N}} |U_i|^s \leq \mathcal{H}^s(X) + \varepsilon.$$

Let $t > s$, then

$$\mathcal{H}^t_\delta(X) \leq \sum_{i \in \mathbb{N}} |U_i|^t = \sum_{i \in \mathbb{N}} |U_i|^{s+(t-s)} = \sum_{i \in \mathbb{N}} |U_i|^s |U_i|^{t-s} \leq \sum_{i \in \mathbb{N}} \delta^{t-s} |U_i|^s \leq \delta^{t-s} \left(\mathcal{H}^s(X) + \varepsilon \right).$$

Letting $\delta \to 0$ gives the required $\mathcal{H}^t(X) = 0$, which proves cases (1) to (3). □

While we always have that the Hausdorff content is a lower bound to the Hausdorff measure, the content is zero precisely when the measure is zero.

Proposition 2.6. *Let (X, d) be a metric space. The Hausdorff measure and content satisfy:*

$$\mathcal{H}^s(X) = 0 \Leftrightarrow \mathcal{H}^s_\infty(X) = 0.$$

Proof. Since $\mathcal{H}^s_\infty(X) < \mathcal{H}^s(X)$ we only need to show that $\mathcal{H}^s_\infty(X) = 0$ implies zero Hausdorff measure. But zero Hausdorff content means that for every $\varepsilon > 0$ there exists a countable cover $\{U_i\}_{i \in \mathbb{N}}$ such that

$$\sum_{i \in \mathbb{N}} |U_i|^s < \varepsilon.$$

But then $\{U_i\}$ is an $\varepsilon^{1/s}$-cover and $\mathcal{H}^s_{\varepsilon^{1/s}}(X) = \mathcal{H}^s(X)$. But since $\varepsilon > 0$ was arbitrary, upon taking limits we get $\mathcal{H}^s(X) = 0$. □

Observe further that for bounded spaces X, the Hausdorff content is always finite, since $\mathcal{H}^s_\infty(X) \leq \operatorname{diam}(X)^s$.

From the results above, we see that the Hausdorff measure and content have a “jumping” characteristic, with a point of discontinuity for $\mathcal{H}^s$. This unique value where the measure jumps from ∞ to 0 is known as the Hausdorff dimension of the space X.

Definition 2.7. *Let (X, d) be a metric space. Its Hausdorff dimension $\dim_H$ is given by*

$$\begin{aligned} \dim_H X &= \inf \{s \geq 0 : \mathcal{H}^s(X) = 0\} = \sup \{s \geq 0 : \mathcal{H}^s(X) = \infty\} \\ &= \inf \{s \geq 0 : \mathcal{H}^s_\infty(X) = 0\} = \sup \{s \geq 0 : \mathcal{H}^s_\infty(X) > 0\}. \end{aligned}$$

2.1 Computing the Hausdorff measure and content

It is generally very difficult to precisely compute the Hausdorff measure and content at the critical component and we often have to contend ourselves with approximations. Most often we will only require knowledge of whether the Hausdorff measure is positive and finite.

To give upper bounds, the most effective approach is to explicitly construct coverings.

Examples We will consider the unit line $L = [0, 1]$ (as a subset of $\mathbb{R}$), the unit circle $\mathbb{S}^1 = \{x : |x| = 1\}$ (as a subset of $\mathbb{R}^2$), and the Cantor set

$$\mathcal{C} = \left\{ x \in [0, 1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, a_i \in \{0, 2\}. \right\}$$

(as a subset of $\mathbb{R}$). We give upper bounds to the Hausdorff content and measure. Since all sets are bounded, we can obtain the (perhaps trivial) open covers of $(-\varepsilon, 1+\varepsilon) \supset L$ for all $\varepsilon > 0$. This gives $\mathcal{H}^s_\infty(L) \leq (1+2\varepsilon)^s \to 1$ as $\varepsilon \to 0$. Hence $\mathcal{H}^s_\infty(L) \leq 1$.

Similarly, we can cover $\mathbb{S}^1$ with the open disc $\{x : |x| < 1+\varepsilon\}$ giving $\mathcal{H}^s_\infty(\mathbb{S}^1) \leq (2+2\varepsilon)^s \to 2^s$ as $\varepsilon \to 0$. Note that this gives $\mathcal{H}^1_\infty(\mathbb{S}^1) \leq 2 < 2\pi = \mathcal{L}^1(\mathbb{S}^1)$.

Finally, since $\mathcal{C} \subset L$, we get the bound $\mathcal{H}^s(\mathcal{C}) \leq 1$.

To give an upper bound for the Hausdorff measure we will need a family of covers whose diameter goes to zero. Very heuristically, this is achieved by taking n intervals of length $(1+\varepsilon)/n$ for L and taking πn balls centred on $\mathbb{S}^1$ of radius $1/n$ for $\mathbb{S}^1$. For $s = 1$ these give $\mathcal{H}^1(L) \leq 1$ and $\mathcal{H}^1(\mathbb{S}^1) \leq 2\pi$ as one would expect.

For the Cantor set, we can cover it by using the construction intervals. That is, it can be covered by 2^n intervals of length $1/3^n$. Hence

$$\mathcal{H}^s_{1/3^n}(\mathcal{C}) \leq 2^n/3^{sn} = (2/3^s)^n.$$

For $s = \log 3/\log 2$, this gives an upper bound of $\mathcal{H}^s(\mathcal{C}) \leq 1$. For small s the expression diverges to ∞, for large s it goes to zero.

Obtaining good lower bounds is made much more difficult by having to consider all possible coverings. Instead we will prove and use the following powerful result, known as the mass distribution principle.

Lemma 2.8 (Mass distribution principle)**.** *Let (X,d) be a metric space, let $s > 0$ and let $E \subset X$ be a bounded non-empty set. Let μ be a finite and positive Borel measure*[3] *supported on E which satisfies*

$$\mu\left(B\left(x,r\right)\right) \leq Cr^s$$

for some universal $C > 0$ and all $x \in E$ and $r > 0$.

Then, $\mathcal{H}^s_\infty(E) \geq \mu(E)/C$ and so $\dim_H E \geq s$.

Proof. Let $\{U_i\}_{i\in\mathbb{N}}$ be a cover of E. Since E is bounded we can also assume $|U_i| < \infty$. Let $x \in U_i \cap E$ and set $r_i = |U_i|$. Then, $U_i \subseteq B(x_i, r_i)$ and

$$\mu(U_i) \subseteq \mu(B(x_i, r_i)) \leq Cr_i^s = C|U_i|^s \tag{2.1}$$

by monotonicity and our assumption on the size of μ. Then

$$\sum_{i\in\mathbb{N}} |U_i|^s \geq \sum_{i\in\mathbb{N}} \frac{\mu(U_i)}{C} \geq \frac{1}{C}\mu\left(\bigcup_{i\in\mathbb{N}} U_i\right) \geq \frac{\mu(E)}{C}$$

by (2.1) and subadditivity of μ. Since the covers were arbitrary, taking the infimum over all covers gives the required result. □

Remark The estimate above may be improved if every U_i is contained in a ball $B(x_i, r_i)$ of radius less than the diameter of U_i. For example, for convex bounded subsets $E \subset \mathbb{R}^d$, each U_i is actually contained in a ball of radius $|U_i|/2$. In this setting, the bound on the Hausdorff content maybe be improved to $\mathcal{H}^s_\infty(E) \geq (2^s/C)\mu(E)$. As we shall see both of these bounds are "optimal".

[3] We do not use the Borel property in the proof here and only use the (outer) measure properties of μ. The reason that it appears is because of a converse to this lemma that we will see later.

Examples For the line L we take μ to be the Lebesgue measure restricted to L. Then, $\mu(B(x,r)) \leq 2r^s$ for all $x \in L$ and $r > 0$, giving $C = 2$. Using the MDP gives $\mathcal{H}^1_\infty(L) \geq \mu(L)/C = \frac{1}{2}$. However, since L is convex we may also use the "improved" version, giving $\mathcal{H}^1_\infty(L) \geq 2^1\mu(L)/2 = 1$ and thus $\mathcal{H}^1(L) = \mathcal{H}^1_\infty(L) = 1$.

For the unit circle we can use μ to be the one dimensional Lebesgue measure supported on the circle, (i.e. the arclength). Considering balls centred on $\mathbb{S}^1$ one can show that $\mu(B(x,r)) = 4\arcsin(r/2)$ for $0 < r \leq 2$ (details omitted). Considering $4/r\arcsin(r/2)$ we can derive that $C = \pi$ is optimal. The MDP gives $\mathcal{H}^1_\infty(\mathbb{S}^1) \geq \mu(\mathbb{S}^1)/C = 2\pi/\pi = 2$ giving $\mathcal{H}^1_\infty(\mathbb{S}^1) = 2$ when combined with our earlier upper bound. This shows that the constant in the MDP is optimal for generic sets.

To show that the Hausdorff measure is 2π we can split the circle into disjoint arcs and use the MDP on each of those circles (with its most optimal bound). Details are omitted, but do give it a try!

Finally we consider the Cantor set $\mathcal{C}$. We construct a measure μ iteratively, giving each construction interval at construction step n measure $1/2^n$. (Details in class) This construction gives a Borel probability measure on $\mathcal{C}$, details are left as an exercise. Now let $x \in \mathcal{C}$, $s = \log 2/\log 3$ and $r > 0$. Without loss of generality we may assume $r < 1/2$ (as otherwise $\mathcal{C} \subset B(x,r)$). Let n be such that

$$3^{-(n+1)} \leq r < 3^{-n}.$$

Then $B(x,r)$ is contained in a level n construction element which has weight $1/2^n$,

$$\mu(B(x,r)) \leq 2^{-n} = (3^s)^{-n} = (3^{-n})^s = 3^s r^s = 2r^2.$$

Using the MDP with $C = 2$ gives $\mathcal{H}^s_\infty(\mathcal{C}) \geq \mu(\mathcal{C})/C = 1/2$.

Exercise 2.1. *Show that the set function μ constructed in the example is a Borel measure.*

Exercise 2.2. *Show, using an appropriate strengthened form of the MDP that the Hausdorff content and measure of the Cantor set for $s = \log 2/\log 3$ is actually equal to 1.*

Exercise 2.3. *Give examples of compact sets such that $\mathcal{H}^s(E) = \infty$ and $\mathcal{H}^s(E) = 0$, where s is the Hausdorff dimension of E.*

The Hausdorff is easily seen to be invariant under isometries as it is defined completely in terms of distances. However, we have yet to prove the "proper" scaling invariance.

Proposition 2.9. *Let (X,d), be a metric space with subsets $E,F \subseteq X$. Let $f : E \to F$ be a bijection such that*

$$d(f(x),f(y)) = c \cdot d(x,y)$$

for all $x,y \in E$ and some fixed $c > 0$. Then $\mathcal{H}^s(F) = c^s\,\mathcal{H}^s(E)$.

Exercise 2.4. *Prove Proposition 2.9.*

We have shown that the value of the Hausdorff content and measure are related (recall positivity). However, if we know that a set F has equal measure and content, we can say more.

Theorem 2.10. *Let (X,d) be a metric space and let $F \subseteq X$ be a $\mathcal{H}^s$-measurable set with $\mathcal{H}^s(F) = \mathcal{H}^s_\infty(F) < \infty$. Then, for all $\mathcal{H}^s$ measurable $E \subseteq F$, we have $\mathcal{H}^s(E) = \mathcal{H}^s_\infty(E)$.*

Proof. By measurability

$$\mathcal{H}^s(E) = \mathcal{H}^s(F) - \mathcal{H}^s(F \setminus E) \le \mathcal{H}^s_\infty(F) - \mathcal{H}^s_\infty(F \setminus E).$$

But $F = E \cup (F \setminus E)$, and so

$$\mathcal{H}^s_\infty(F) \le \mathcal{H}^s_\infty(E) + \mathcal{H}^s_\infty(F \setminus E)$$

and $\mathcal{H}^s(E) \le \mathcal{H}_\infty(E)$, completing the proof. □

2.2 More general Hausdorff measures (bonus?)

The choice of looking at the diameter of sets raised to a power, i.e. the $|U_i|^s$ appearing in the Hausdorff measure is inspired by the "normal" geometric behaviour that sets in $\mathbb{R}^d$ exhibit. However, we need not have restricted ourselves to this. Given any increasing $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ we can define

$$\mathcal{H}^g_\delta(X) = \inf \left\{ \sum_{i \in \mathbb{N}} g(|U_i|) : U_i \text{ is a } \delta\text{-cover of } X \right\}$$

letting $\mathcal{H}^g$ and $\mathcal{H}^g_\infty$ be the natural limits in δ.

These general Hausdorff measures and contents can be useful in more abstract metric spaces (such as those arising in random geometry) and differentiate behaviour for decay behaviour that is "slower" than polynomial.

Many results extend to these general Hausdorff measures, and there is a natural analogue of the mass distribution principle.

3 Covering Lemmas

Clearly, coverings are important to understand the Hausdorff measure and content of a space. But even more generally they are crucial to understanding metric spaces and allow us to analyse structures and extrapolate notions such as differentiability to "rough" metric spaces. We will get to see several covering lemmas in this section, the simplest being this finite covering lemma.

Lemma 3.1 (Finite Vitali Covering Lemma)**.** *Let $\mathcal{B} = \{B_i\}$ be a finite collection of closed balls in (X, d). There exists a subcollection $\mathcal{B}' = \{B_j\} \subseteq \mathcal{B}$ such that all $B(x, r) \in \mathcal{B}'$ are mutually disjoint and*

$$\bigcup \mathcal{B} \subseteq \bigcup 3\,\mathcal{B}' = \bigcup_{B(x_j, r_j) \in \mathcal{B}'} B(x_j, 3r_j).$$

Proof. The proof is constructive. Let j_1 be such that $B_{j_1} = B(x_{j_1}, r_{j_1})$ has the largest of all radii in $\mathcal{B}$ choosing arbitrarily if there is more than one. By induction we choose a disjoint collection of balls. Assuming we have found a disjoint collection of balls $B_{j_1} \cup B_{j_2} \cup \dots B_{j_k}$, we choose $B_{j_{k+1}}$ to be the largest ball in $\mathcal{B}$ that is disjoint from $B_{j_1} \cup \dots \cup B_{j_k}$. We terminate the process once there is no such ball left.

To show that the enlargement contains $\bigcup \mathcal{B}$, consider an arbitrary ball $B_i \in \mathcal{B}$. If $B_i \in \mathcal{B}'$ we are done, so assume the contrary. But then B_i must intersect a ball $B_j \in \mathcal{B}'$ with no smaller radius as otherwise B_i would be a member of $\mathcal{B}'$. Hence $B_i \cap B_j \neq \varnothing$ and the triangle inequality implies that $B_i \subset 3B_j$. This proves the lemma. □

The proof for arbitrary collections is similar, but requires the axiom of choice (i.e. well-orderings).

Lemma 3.2 (Vitali $5r$-Covering Lemma)**.** *Let $\mathcal{B}$ be an arbitrary collection of balls in a metric space (X, d) with diameter uniformly bounded above. Then there exists a disjoint subcollection $\mathcal{B}'$ such that for every $B \in \mathcal{B}$ there exists $B' \in \mathcal{B}'$ with $B \subset 5B'$.*

Proof. We partition $\mathcal{B}$ by size of balls and write $\mathcal{B}_n = \{B(x, r) \in \mathcal{B} : 2^{-n} < r \leq 2^{-n+1}\}$ for $n \in \mathbb{Z}$. By the boundedness of the balls there exists $N \in \mathbb{Z}$ such that $\mathcal{B}_n = \varnothing$ for all $n < N$ and $\mathcal{B}_N \neq \varnothing$.

We define $\mathcal{B}'$ inductively. Set $\mathcal{A}_0 = \mathcal{B}_N$ and let $\mathcal{B}'_0$ be a maximal disjoint subcollection of $\mathcal{A}_0$ (this requires the Axiom of choice). Having defined $\mathcal{A}_n$ and $\mathcal{B}_n$, we define

$$\mathcal{A}_{n+1} = \{B \in \mathcal{B}_{n+1} : B \cap B' = \varnothing \text{ for all } B' \in \mathcal{B}'_0 \cup \cdots \cup \mathcal{B}'_n\}$$

and let $\mathcal{B}'_{n+1}$ be a maximal disjoint subcollection of $\mathcal{A}_{n+1}$.

Let $\mathcal{B}' = \bigcup_{n \in \mathbb{N}_0} \mathcal{B}'_n$. It remains to show that $\mathcal{B}'$ satisfies the assumptions. Clearly, by construction, $\mathcal{B}'$ is a disjoint family of balls. Consider an arbitrary ball $B \in \mathcal{B}$. There exists n such that $B \in \mathcal{B}_n$ and we may assume B is not contained in $\mathcal{B}'_n$ as otherwise there is nothing to prove. There are two cases to consider: either $B \notin \mathcal{A}_n$, in which case there exists $B' \in \mathcal{B}'_0 \cup \cdots \cup \mathcal{B}'_{n-1}$ such that $B \cap B' \neq \varnothing$ and as $\operatorname{diam} B' > \operatorname{diam} B$ we have $B \subset 3B'$. The other case happens when $B \in \mathcal{A}_n$ where B is part of the new collection of smaller balls but is not in the maximal disjoint subset and there exists $B' \in \mathcal{B}'_n$ such that $B \cap B' \neq \varnothing$. Since $\operatorname{diam} B' < 2 \operatorname{diam} B$, the triangle inequality gives $B \subset 5B'$. This proves our claim. □

Remark. If (X, d) is separable, the cover can be taken to be countable. In fact for metric spaces, the following are equivalent notions: X is separable, X is second countable, X is hereditarily Lindelöf.

Remark. For bounded doubling spaces, we may assume that each of the $\mathcal{B}_n$ are finite as the doubling property implies that we will eventually "run out of space" for too many disjoint balls of the same size. Since $\mathbb{R}^d$ is doubling, we can for bounded subsets assume that the countably many balls in $\mathcal{B}'$ can be sorted in decreasing order.

Remark. The balls can also be replaced by "reasonable" closed and bounded subsets of X, such as convex sets, defining their $5r$ enlargements in a natural way..

3.1 Vitali's covering theorem for the Hausdorff measure

We can use the $5r$ covers mentioned above to create a countable cover that approximates a set up to negligible Hausdorff measure. We first define the notion of a Vitali cover.

Definition 3.3. *Let $E \subseteq X$ for some metric space (X, d). Assume $\mathcal{V}$ is a family of sets such that*

$$\sup_{x \in A} \inf \{|B| : x \in B \in \mathcal{V}\} = 0. \tag{3.1}$$

We say that $\mathcal{V}$ is a Vitali cover *of A. If $\mathcal{V}$ is a collection of open/closed balls in X, we say $\mathcal{V}$ is an open/closed Vitali cover of A.*

Theorem 3.4. *Let $s > 0$ and let $\mathcal{V}$ be a closed Vitali cover of $E \subseteq \mathbb{R}^d$. Then there exists a countable, disjoint subcollection $\mathcal{V}' \subseteq \mathcal{V}$ such that either*

$$\mathcal{H}^s\left(E \setminus \bigcup_{B_i \in \mathcal{V}'} B_i\right) = 0 \tag{3.2}$$

or

$$\sum_{B_i \in \mathcal{V}'} |B_i|^s = \infty. \tag{3.3}$$

Proof. First assume E is bounded. Let $\mathcal{V}' = \{B_i\}$ be the disjoint $5r$ cover constructed in Vitali's covering lemma. If $\mathcal{V}'$ is finite, we must necessarily have $F \subset \bigcup_{B_i} B_i$ and (3.2) is immediately satisfied. Hence we may assume $\mathcal{V}'$ is countably infinite and B_i can be indexed by $i \in \mathbb{N}$. Further, since $E \subset \mathbb{R}^d$ is bounded we may assume that $|B_i|$ is decreasing in i. We may also assume that $\sum_{i=1}^{\infty} |B_i|^s < \infty$ as otherwise (3.3) applies.

Let $k \in \mathbb{N}$ and note that $\mathbb{R}^d \setminus \bigcup_{i=1}^{k} B_i$ is open. Thus, for all $x \in \mathbb{R}^d \setminus \bigcup_{i=1}^{k} B_i$ there must exist a ball $B \in \mathcal{V}$ that contains x, and is of diameter at most that of B_k. Now B must intersect at least one ball $B_j \in \mathcal{V}'$ of diameter in-between that of B_k and B (as otherwise it would have been picked in $\mathcal{V}'$). Hence $j > k$ and

$$E \setminus \bigcup_{i=1}^{k} B_i \subseteq \bigcup_{i=k+1}^{\infty} B_i.$$

Temporarily fix $\delta > 0$ and chose k such that $|5B_i| < \delta$ for all $i \geq k$. Clearly, $k \to \infty$ as $\delta \to 0$. Then, putting it all together,

$$\begin{aligned}
\mathcal{H}^s_\delta\left(E \setminus \bigcup_{i=1}^{\infty} B_i\right) &\leq \mathcal{H}^s_\delta\left(E \setminus \bigcup_{i=1}^{k} B_i\right) \\
&\leq \mathcal{H}^s_\delta\left(\bigcup_{i=k+1}^{\infty} B_i\right) \\
&\leq \sum_{i=k+1}^{\infty} 5^s |B_i|^s
\end{aligned}$$

Since the sum is finite, letting $\delta \to 0$ gives $\mathcal{H}^s\left(E \setminus \bigcup_{i=1}^{\infty} B_i\right)$=0, as required. □

Proposition 3.5. *Let $E \subseteq \mathbb{R}^d$ be a Borel set. Let μ be a finite Borel measure on $\mathbb{R}^d$ and let $0 < c < \infty$.*

1. *If $\limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} \leq c$ for all $x \in E$ then $\mathcal{H}^s(E) \geq \mu(E)/c$.*
2. *If $\limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} \geq c$ for all $x \in E$ then $\mathcal{H}^s(E) \leq 2^s \mu(\mathbb{R}^d)/c$.*

Proof. Statement (1.) follows from the mass distribution principle since $\mu(B(x,r)) \leq (c + \varepsilon) r^s$ holds for all $\varepsilon > 0$ for small enough $r > 0$. Details are left as an exercise, see below.

Proof of statement (2.). Let $\varepsilon, \delta > 0$ and let $\mathcal{V}$ be the collection of closed balls $B(x,r)$ for $x \in E$ and $0 < r < \delta$ such that $\mu(B(x,r)) \geq (c - \varepsilon) r^s$. It can easily be checked that $\mathcal{V}$

is a closed Vitali cover of E for all δ. Hence, using Vitali's covering theorem, there exists $\mathcal{V}' \subset \mathcal{V}$ which is countable and pairwise disjoint such that

$$\mathcal{H}^s\left(E \setminus \bigcup_{B\in\mathcal{V}'} B\right) = 0 \quad \text{or} \quad \sum_{B\in\mathcal{V}'} |B|^s = \infty.$$

We divide into two cases and first consider the diverging sum case:

$$\begin{aligned}
\infty = \sum_{B\in\mathcal{V}'} |B|^s &= 2^s \sum_{B(x,r)\in\mathcal{V}'} r^s \\
&\leq \frac{2^s}{c-\varepsilon} \sum_{B(x,r)\in\mathcal{V}'} \mu\left(B(x,r)\right) && \text{(by definition of } \mathcal{V}) \\
&= \frac{2^s}{c-\varepsilon}\mu\left(\bigcup_{B\in\mathcal{V}'} B\right) && \text{(by additivity of Borel measures)} \\
&\leq \frac{2^s}{c-\varepsilon}\mu\left(\mathbb{R}^d\right)
\end{aligned}$$

and so $\mu\left(\mathbb{R}^d\right) = \infty$ and the conclusion holds trivially.

The second case is similar, noting that $\mathcal{V}'$ depends on the choice of ε and δ.

$$\begin{aligned}
\mathcal{H}^s(E) &= \mathcal{H}^s\left(E \setminus \bigcup_{B\in\mathcal{V}'} B\right) + \mathcal{H}^s\left(E \cap \bigcup_{B\in\mathcal{V}'} B\right) \\
&= \mathcal{H}^s\left(E \cap \bigcup_{B\in\mathcal{V}'} B\right) && \text{(since the first term is 0)} \\
&\leq \liminf_{\delta\to 0} \mathcal{H}^s_\delta\left(\bigcup_{B\in\mathcal{V}'} B\right) \leq \liminf_{r\to 0} \sum_{B(x,r)\in\mathcal{V}'} 2^s r^s \\
&\leq \liminf_{r\to 0} \frac{2^s}{c-\varepsilon}\cdot\mu\left(\bigcup_{B\in\mathcal{V}'} B\right) \leq \frac{2}{c-\varepsilon}\mu\left(\mathbb{R}^d\right) \qquad \square
\end{aligned}$$

Exercise 3.1. *Prove Statement (1.) in Proposition 3.5.*

3.2 A Vitali covering theorem for Radon measures

One can also give a Vitali covering theorem for Radon measures which, unlike the Hausdorff measure, are finite for all compact sets. We will not prove the following statements, but note that they are important in the study of geometric measures and generalise (to some extend) the last couple of statements.

Theorem 3.6. *Let μ be a Radon measure of $\mathbb{R}^d$. Let $A \subseteq \mathbb{R}^d$ and $\mathcal{V}$ be a closed Vitali cover of A. Then there exists a subcollection $\mathcal{V}' \subset \mathcal{V}$ of disjoint balls such that*

$$\mu\left(A \setminus \bigcup_{B\in\mathcal{V}'} B\right) = 0.$$

Proof. A proof can be found in [1, Theorem 2.8]. $\square$

Similarly we can study measures relative to each other by looking at local measure ratios. This is in analogy to Proposition 3.5.

Proposition 3.7. *Let μ and ν be Radon measures on $\mathbb{R}^d$. Let $0 < c < \infty$ and $A \subseteq \mathbb{R}^d$.*

1. *If $\liminf_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \le c$ for all $x \in A$, then $\mu(A) \le c\nu(A)$.*
2. *If $\limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \ge c$ for all $x \in A$, then $\mu(A) \ge c\nu(A)$.*

4 Cookie Cutters

In this section we will study a simple dynamical system which has a "fractal-like" invariant set, but is not linear as the earlier examples were. That is, the invariant set is not self-similar in its strictest sense.

The system we consider acts upon a closed interval $X \subset \mathbb{R}$. Let X be a non-empty closed interval and let X_1 and X_2 be disjoint subintervals of X. Let f be a dynamic such that $f : X_1 \cup X_2 \to X$ such that $f|_{X_1}$ and $f|_{X_2}$ are both bijective. We will further assume that f is twice differentiable and that $|f'(x)| > 1$ for all $x \in X_1 \cup X_2$. Note that the closedness implies the crucial fact that the derivative is bounded away from 1 uniformly on $X_1 \cup X_2$. We write f^k for the k-th iterate of f.

Consider the points that never leave X_1 and X_2. That is, the points

$$E = \left\{x \in X : f^k(x) \in X_1 \cup X_2 \text{ for all } k \in \mathbb{N}\right\}.$$

The set is invariant under f, i.e. $E = f(E) = f^{-1}(E)$ since $x \in E$ if and only if $f(x) \in E$. Similarly, any $x \in X \setminus E$ must, by definition, be eventually be mapped outside of X_1 and X_2. As such, E is the "largest" invariant set. Since it is contained in the preimages $f^{-k}(X)$, which is a decreasing collection of compact sets, E mus be compact itself.

We can equivalently see the set arising as an attractor of an iterated function system. Let $F_1, F_2, \ldots, F_N$ be a finite collection of strict contractions on $\mathbb{R}^d$. Then there exists a unique, compact subset $X \subset \mathbb{R}^d$ such that

$$X = \bigcup_{i=1}^{N} F_i(X)$$

called the attractor of the iterated function system (IFS) $\{F_1, \ldots, F_N\}$. This fact holds, because $X \mapsto \bigcup_{i=1}^N F_i(X)$ is a contractive map on the metric space $K(\mathbb{R}^d), d_H)$, the space of all compact subsets of $\mathbb{R}^d$ with the Hausdorff metric. An application of the Banach fixed point theorem shows that there is a unique point in $K(X)$ (i.e. a unique compact subset) that is fixed by this operation.

In our case, the contracting maps are given by the two inverse branches of f, that is the inverses $F_1 = (f|_{X_1})^{-1}$ and $F_2 = (f|_{X_2})^{-1}$. Since E is a compact set invariant under f, it must also be invariant under the IFS $\{F_1, F_2\}$ and hence it is the unique attractor.

We will index the images of X under the IFS by sequences of 1s and 2s. We write $I_k = \{1, 2\}^k$ for all sequences of length k. For sequences $\mathtt{i} \in I_k$ of length k, we write

$$X_{\mathtt{i}} = X_{i_1 i_2 \ldots i_k} = F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_k}(X) = F_{\mathtt{i}}(X).$$

We may further have need to refer to all finite words, which we denote by $I_* = \{1, 2\}^* = \bigcup_{k=0}^{\infty} I_k$, where I_0 consists only of the empty sequence. We can iterate the construction and see that $X_{\mathtt{i}} \supseteq X_{\mathtt{i},1} \cup X_{\mathtt{i},2}$. In fact, we can approximate the attractor E by their images under F_1 and F_2. We define $E_k = \bigcup_{\mathtt{i} \in I_k} X_{\mathtt{i}}$ which is the union of 2^k disjoint closed intervals. Clearly $E \subseteq E_k \subseteq E_{k-1}$ for all k and it can be shown that $E_k \to E$ in the Hausdorff metric sense.

A simple example of a dynamical system is $f(x) = 3x \mod 1$, where $X_1 = [0, 1/3]$ and $X_2 = [2/3, 1]$. Equivalently, $F_1(x) = x/3$ and $F_2(x) = x/3 + 2/3$ with $X = [0, 1]$. The arising set is the Cantor middle third set, which we have seen before. However the construction is much more flexible and we can use nonlinear maps such as $F_1(x) = x/3 + \sqrt{x}/10$ and $F_2(x) = x/3 + \sin(x)/20 + \exp(x/20)/10 + 2/3$ with $X = [0, 1]$, as long as the conditions of contractiveness (or expansiveness for f) and differentiability is guaranteed.

4.1 Bounded distortion

Bounded distortion makes the notion of "self-similar up to some distortion" more concrete by stating that small neighbourhoods are uniformly bounded away from an exact similitude. We will first show that concept of bounded distortion in a general way before applying it to the cookie cutters.

Let $\phi : X_1 \cup X_2 \to \mathbb{R}$ be a Lipschitz function, satisfying

$$|\phi(x) - \phi(y)| \leq c|x - y|$$

for some $c > 0$. We are interested in evaluating ϕ at the iterates of a point x and we define

$$S_k\phi(x) = \phi(x) + \phi(f(x)) + \phi(f^2(x)) + \cdots + \phi(f^{k-1}(x)) = \sum_{j=0}^{k-1} \phi(f^j(x)).$$

This is certainly defined whenever $x \in E$, but we may also more generally assume that $x \in X_{\mathtt{i}}$ for some $\mathtt{i} \in I_k$.

The principle of bounded variation states that $S_k\phi(x)$ does not vary much with x in a uniform sense.

Proposition 4.1 (Principle of bounded variation). *Let $\phi : X \to \mathbb{R}$ be a Lipschitz function.*

1. *There exists a number b such that for all $k \in \mathbb{N}$ and all $\mathtt{i} \in I_k$ we have*

$$|S_k\phi(x) - S_k\phi(y)| \leq b$$

for all $x, y \in X_{\mathtt{i}}$.

2. *More generally, for all $q \geq k$ and all $\mathtt{i} \in I_q$ we have*

$$|S_k\phi(x) - S_k\phi(y)| \leq b|X|^{-1}|X_{i_{k+1}\dots i_q}|$$

for all $x, y \in X_{\mathtt{i}}$.

Proof. Repeatedly applying the inverses F_i, we have, for all $\mathtt{i} \in I_k$.

$$|X_{\mathtt{i}}| = |F_{i_1} \circ \cdots \circ F_{i_k}(X)| \leq c_{\max}^k |X|,$$

where $c_{\max} = \max_{j \in \{1,2\}} \max_{x \in X} |F_i'(x)| < 1$. Now if $x, y \in X_{\mathtt{i}}$ then $f^j x, f^j y \in X_{i_{j+1}\dots i_k}$ and

$$|\phi(f^j x) - \phi(f^j y)| \leq c|f^j x - f^j y| \leq c|X_{i_{j+1}\dots i_k}| \leq c \cdot c_{\max}^k |X|.$$

Thus,

$$
\begin{aligned}
|S_k\phi(x) - S_k\phi(y)| &= \left|\sum_{j=0}^{k-1}\phi(f^j x) - \sum_{j=0}^{k-1}\phi(f^j y)\right| \\
&\le \sum_{j=0}^{k-1}|\phi(f^j x) - \phi(f^j y)| \\
&\le \sum_{j=0}^{k-1} c\cdot c_{\max}^{k-j}|X| \le c\cdot c_{\max}|X|/(1-c_{\max})
\end{aligned}
$$

which proves the first statement with $b = c\cdot c_{\max}|X|/(1-c_{\max})$.

The second statement is similar and follows upon noting that for $x, y \in X_{\mathtt{i}}$, then $f^j x, f^j y \in X_{i_{j+1}\dots i_q}$ and

$$|\phi(f^j x) - \phi(f^j y)| \le c\cdot c_{\max}^{k-j}|X_{i_{k+1}\dots i_q}|. \qquad \square$$

An alternative form of the first statement is

$$e^{-b} \le \frac{\exp(S_k\phi(x))}{\exp(S_k\phi(y))} \le e^b$$

which will come in useful later.

We will now consider the special function $\phi(x) = -\log|f'(x)|$. It is easy to verify that ϕ is Lipschitz on $X_1 \cup X_2$. The function is chosen to represent the geometric size of subsets of X. Using the chain rule repeatedly we obtain

$$(f^k)'(x) = f'(f^{k-1}x)\cdot f'(f^{k-2}x)\dots f'(x)$$

and taking logarithms we obtain

$$-\log|(f^k)'(x)| = \sum_{j=0}^{k-1} -\log|f'(f^j x)| = \sum_{j=0}^{k-1}\phi(f^j x) = S_k\phi(x).$$

The mapping $f^k : X_{\mathtt{i}} \to X$ for $\mathtt{i} \in I_k$ is a bijection with the additional property that it is bi-Lipschitz with constant close to $|X_{\mathtt{i}}|$. This is encapsulated in the principle of bounded distortion.

Proposition 4.2 (Principle of bounded distortion)**.** *There exist constants b_0 and b_1 such that for all $\mathtt{i} \in I_k$ and $k \in \mathbb{N}$ we have*

$$b_0^{-1} \le |X_{\mathtt{i}}||(f^k)'(x)| \le b_0$$

for all $x \in X_{\mathtt{i}}$. Moreover $f^k : X_{\mathtt{i}} \to X$ satisfies

$$b_1^{-1}|y-z| \le |f^k(y) - f^k(z)||X_{\mathtt{i}}| \le b_1|y-z|$$

for all $y, z \in X_{\mathtt{i}}$.

Proof. Recall that $X_{\mathtt{i}} = F_{i_1}\circ\dots\circ F_{i_k}(X)$ and so $f^k : X_{\mathtt{i}} \to X$ is a twice differentiable bijection. The mean value theorem gives that for every $x, y \in X_{\mathtt{i}}$ there exists $z \in X_{\mathtt{i}}$ such that

$$f^k(x) - f^k(y) = (x-y)(f^k)'(z).$$

We may chose x, y to be the endpoints of $X_{\mathtt{i}}$. Then

$$|X| = |X_{\mathtt{i}}||(f^k)'(z)|$$

for some $z \in X_{\mathtt{i}}$. The bounded variation principle further gives

$$e^{-b} \leq \frac{|(f^k)'(u)|}{|(f^k)'(w)} \leq e^b$$

for all $u, w \in X_{\mathtt{i}}$. Combining these estimates proves our statements above. □

We may also need to you the equivalent form in terms of the inverse branches of f^k,

$$b_0^{-1} \leq \frac{|X_{\mathtt{i}}|}{|(F_{i_1} \circ \cdots \circ F_{i_k})'(x)|} \leq b_0.$$

Note that in the special case of similarity mappings the contractions have constant derivative and

$$|X_{\mathtt{i}}| = c_{i_1} c_{i_2} \ldots c_{i_k} |X|,$$

where $c_i = F_i'$ is the contraction ratio of the similarity F_i.

Recall that we needed separation in Caratheodory's criterion and we can prove a separation condition using the last proposition

Corollary 4.3. *Let E be a cookie-cutter set and let $d = \operatorname{dist}(X_1, X_2)$ be the distance between the two disjoint intervals.*

1. *For all $\mathtt{i} \in I_k$,*
$$d b_1^{-1} |X_{\mathtt{i}}| \leq \operatorname{dist}(X_{\mathtt{i},1}, X_{\mathtt{i},2}) \leq |X_{\mathtt{i}}|.$$
2. *Let $\lambda = d b_1^{-1} c_{\min}$. For all $\mathtt{i} \in I_k$, if $x \in X_{\mathtt{i}} \cap E$ and $|X_{\mathtt{i}}| \leq r < |X_{\mathtt{i}}| c_{\min}^{-1}$, then*
$$B(x, \lambda r) \cap E \subseteq X_{\mathtt{i}} \cap E \subseteq B(x, r).$$

Proof. Left as an exercise. □

Exercise 4.1. *Prove the corollary above.*

4.2 "Almost" self-similar sets

We can use the principle of bounded distortion to prove forms of self-similarity that will be useful later.

Corollary 4.4. *Let E be a cookie-cutter set. Then there are $c, r_0 > 0$ such that for all $B = B(x, r)$ with $x \in E$ and $0 < r < r_0$ there exists mapping $g : E \cap B \to E$ with*

$$c^{-1} r^{-1} |x - y| \leq |g(x) - g(y)| \leq c r^{-1} |x - y| \quad \textit{for all } x, y \in E \cap B.$$

Heuristically, this means that every ball of E is a not too small and not too distorted subset of E itself.

The following corollary can be considered a 'dual' to the one above. It essentially states that every ball in E also contains a not too small and not too distorted copy of E.

Corollary 4.5. *Let E be a cookie cutter set. Then there are $c, r_0 > 0$ such that for all $B = B(x,r)$ with $x \in E$ and $0 < r < r_0$ there exists a mapping $g : E \to E \cap B$ with*

$$c^{-1}r|x-y| \le |g(x)-g(y)| \le cr|x-y|.$$

As it turns out, the latter property will be more useful as an assumption for general sets. The former property is essentially too restrictive as every ball needs to be contained in E (after rescaling) but the latter condition means that the set is only contained in every ball, which does not exclude the set to have more points.

Exercise 4.2. *Prove the corollaries above.*

5 Thermodynamic formalism and Pressure

Recall that we could deduce the dimension formula for the Cantor middle-third set from it "linearity". More generally, we say that a compact, non-empty subset of $\mathbb{R}^d$ is **self-similar** if it satisfies

$$E = \bigcup_{i=1}^{n} f_i(E)$$

for some similarities f_i, i.e. those maps for which $|f_i(x) - f_i(y)| = c_i|x-y|$ for some $c_i \in (0,1)$. Equivalently, $f_i(x) = c_i\mathbf{O}_i x + \mathbf{t}_i$, where $\mathbf{O}_i$ is an orthogonal matrix and $\mathbf{t}_i$ is a translation vector in $\mathbb{R}^d$.

If the images $f_i(E)$ do not overlap, we can guess (as before) that the Hausdorff measure is positive and finite to get an expression for the dimension:

$$\mathcal{H}^s(E) = \sum_{i=1}^{n} \mathcal{H}^s(f_i(E)) = \sum_{i=1}^{n} (c_i)^s\, \mathcal{H}^s(E)$$

and so $\sum_{i=1}^{n}(c_i)^s = 1$. The unique value for which this holds is called the **similarity dimension** of E. The similarity dimension is always an upper bound to the Hausdorff dimension, irrespective of whether images $f_i(E)$ overlap and is sharp whenever the images are pairwise disjoint, see exercise below.

In this section we will develop a systematic approach for the above for non-linear attractors and repellers. We will develop these for cookie-cutter sets only, though the methods will apply in much greater generality.

Exercise 5.1. *Show that the similarity dimension is unique. That is, given any finite, positive number of contractions with contraction ratios $c_i \in (0,1)$, there is a unique solution to*

$$\sum_{i=1}^{n} (c_i)^s = 1$$

Exercise 5.2. *Show that the Hausdorff dimension of a self-similar set is bounded above by the similarity dimension.*

Exercise 5.3. *Let E be a self-similar set with mappings f_i. Assume that $f_i(E) \cap f_j(E) = \varnothing$ for all $i \neq j$. Using the mass distribution principle and a suitable measure, show that the Hausdorff dimension is equal to the similarity dimension.*

5.1 Cookie-cutter thermodynamic formalism

We recall some notation from the previous section. Let X be a real closed interval with disjoint sub intervals X_1 and X_2. Let $f : X_1 \cup X_2 \to X$ be an expanding mapping with continuous second derivative with $f|_{X_1}$ and $f|_{X_2}$ bijective. There exists a repeller E which is also the unique invariant set $E = \bigcup_{i=1,2} F_i(E)$, where $F_i = (f|_{X_i})^{-1}$.

Let φ be a Lipschitz function $\varphi : X_1 \cup X_2 \to \mathbb{R}$. That is,

$$|\varphi(x) - \varphi(y)| < C|x - y|$$

for all $x, y \in X_1 \cup X_2$ and some uniform $C > 0$. The bounded variation theory from last section applies. So, writing

$$S_k\varphi(x) = \sum_{j=0}^{k-1} \varphi(f^j(x)) \quad \text{for } x \in \bigcup_{\mathtt{i}\in I_k} X_{\mathtt{i}}$$

there exists $b > 0$ such that

$$|S_k\varphi(x) - S_k\varphi(y)| \leq b \quad \iff \quad e^{-b} \leq \frac{\exp(S_k\varphi(x))}{\exp(S_k\varphi(y))} \leq e^b$$

for all $x, y \in X_{\mathtt{i}}$, $\mathtt{i} \in I_k$, $k \in \mathbb{N}$.

An appropriate choice of φ will lead us to the dimension formula. However, we will develop the theory for general Lipschitz φ first.

Our next goal is to find a measure μ on E such that

$$\mu(X_{\mathtt{i}}) \sim \exp(S_k\varphi(x)) \exp(-kP(\varphi))$$

for all $\mathtt{i} \in I_k$ and $x \in X_{\mathtt{i}}$. Such a measure (if it exists) is called a Gibbs measure with potential φ. The constant $P(\varphi)$ that only depends on the potential φ is called the (topological) pressure of the potential φ.

Theorem 5.1. *For all $k \in \mathbb{N}$ and $\mathtt{i} \in I_k$, let $x_{\mathtt{i}} \in X_{\mathtt{i}}$. Then the limit*

$$P(\varphi) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\mathtt{i}\in I_k} \exp S_k\varphi(x_{\mathtt{i}})$$

exists and does not depend on the $x_{\mathtt{i}} \in X_{\mathtt{i}}$ chosen.

Further, there exists a Borel probability measure μ that is supported on E and a constant $a_0 > 0$ such that

$$\frac{1}{a_0} \leq \frac{\mu(X_{\mathtt{i}})}{\exp(-kP(\varphi) + S_k\varphi(x))} \leq a_0$$

for all $k \in \mathbb{N}$, $\mathtt{i} \in I_k$, and $x \in X_{\mathtt{i}}$.

Proof. Fix $w \in E$. Note that

$$S_{k+m}\varphi(x) = S_k\varphi(x) + S_m\varphi(f^k(x)).$$

Taking exponentials and summing gives

$$\sum_{x:f^{k+m}x=w} \exp S_{k+m}\varphi(x) = \sum_{x:f^{k+m}(x)=w} \exp(S_k\varphi(x)) \exp(S_m\varphi(f^k(x)))$$

$$
\begin{aligned}
&= \sum_{z:f^m z=w} \sum_{x:f^k x=z} \exp(S_k\varphi(x)) \exp(S_m\varphi(z)) \\
&\le e^b \sum_{z:f^m z=w} \sum_{x:f^k x=w} \exp(S_k\varphi(x)) \exp(S_m\varphi(z)).
\end{aligned}
$$

Writing $s_k = \sum_{x:f^k x=w} \exp(S_k\varphi(x))$, we get $s_{k+m} \le e^b s_k s_m$. Similarly, one can give a lower bound and we see that s_k is quasi-multiplicative:

$$
e^{-b} s_k s_m \le s_{k+m} \le e^b s_k s_m.
$$

Taking logarithms gives

$$
\log s_k + \log s_m - b \le \log s_{k+m} \le \log s_k + \log s_m + b
$$

and using Fekete's lemma (see Lemma 1.10 and Corollary 1.11) there exists

$$
c = \lim_{k\to\infty} \frac{1}{k \log s_k} \in [-\infty, \infty)
$$

satisfying $\log s_k \ge kc - b$. Similarly,

$$
-\log s_{k+m} \le (-\log s_k) + (-\log s_m) + b
$$

giving

$$
-c = \lim_{k\to\infty} \frac{-1}{k} \log s_k \in [-\infty, \infty)
$$

and $-\log s_k \ge -kc - b$. Hence $c \in (-\infty, \infty)$. Note that for this particular choice of $x_{\mathtt{i}}$, $c = P(\varphi)$. The argument above can be sufficiently altered to make the choice of $x_{\mathtt{i}}$ arbitrary, see Exercise 5.4.

Combining the bounds on s_k, we get

$$
e^{-b} \exp(kP(\varphi)) \le s_k \le e^b \exp(kP(\varphi)) \tag{5.1}
$$

We now construct a measure μ by defining discrete measures μ_m and taking their limit. For any $A \subseteq \mathbb{R}$ we define

$$
\mu_m(A) = \frac{1}{s_m} \sum_{x\in A:f^m=w} \exp(S_m\varphi(x))
$$

Equivalently, it is giving each of the 2^m many images $x_{\mathtt{i}} = F_{i_1} \circ \cdots \circ F_{i_m}(w)$ a normalised weight given by $\exp(S_m\varphi(x_{\mathtt{i}}))$. This clearly makes μ_m discrete as well as a probability measure. Hence a subsequence of μ_m converges weakly to some Borel measure μ with support on E, see Theorem 1.8.

For any $X_{\mathtt{i}}$ with $\mathtt{i} \in I_k$ for $k \le m$ we have

$$
\begin{aligned}
\mu_m(X_{\mathtt{i}}) &= \frac{1}{s_m} \sum_{x\in X_{\mathtt{i}}:f^m x=w} \exp(S_m\varphi(x)) \\
&= \frac{1}{s_m} \sum_{x\in X_{\mathtt{i}}:f^m x=w} \exp(S_k\varphi(x)) \exp(S_{m-k}\varphi(f^k(x))).
\end{aligned}
$$

Letting $y \in X_{\mathtt{i}}$ be arbitrary we get

$$
e^{-b}\mu_m(X_{\mathtt{i}}) \quad \le \quad \frac{1}{s_m} \exp(S_k\varphi(y)) \sum_{z\in X:f^{m-k}=w} \exp(S_{m-k}\varphi(z)) \quad \le \quad e^b \mu_m(X_{\mathtt{i}}).
$$

By definition of s_{m-k},

$$e^{-b}\mu_m(X_{\mathtt{i}}) \le \exp(S_k\varphi(y))\frac{s_{m-k}}{s_m} \le e^{b}\mu_m(X_{\mathtt{i}})$$

and as $s_m = s_{k+m-k} \sim s_k s_{m-k}$,

$$e^{-2b}\mu_m(X_{\mathtt{i}}) \le \frac{1}{s_k}\exp(S_k\varphi(y)) \le e^{2b}\mu_m(X_{\mathtt{i}})$$

which holds for all $m \ge k$, so in particular,

$$e^{-2b} \le \frac{s_k\mu(X_{\mathtt{i}})}{\exp(S_k\varphi(y))} \le e^{2b}.$$

Finally we use (5.1) which gives

$$e^{-3b} \le \frac{\mu(X_{\mathtt{i}})}{\exp(S_k\varphi(y) - kP(\varphi))} \le e^{3b}$$

as required. □

Exercise 5.4. *Complete the proof above by showing that the points $x_{\mathtt{i}}$ are indeed arbitrary*

5.1.1 Generalisations

While we have restricted to one-dimensional dynamics with only two expanding maps, the study above can be extended to much larger families of maps.

Topological pressure Recall that the points $x_{\mathtt{i}}$ were chosen arbitrarily. For more general mappings, the expression may however depend on the point chosen. Usually, the points $x_{\mathtt{i}}$ are taken to be the fixed points of the map under iteration. Since for every $\mathtt{i} \in I_k$ there is a unique $x_{\mathtt{i}} \in X_{\mathtt{i}}$ such that

$$F_{i_1} \circ \cdots \circ F_{i_k}(x_{\mathtt{i}}) = x_{\mathtt{i}} \iff f^k(x_{\mathtt{i}}) = x_{\mathtt{i}}$$

we may choose to define the pressure over the 2^k fixed points of f^k,

$$P(\varphi) = \lim_{k\to\infty}\frac{1}{k}\sum_{x\in\mathrm{Fix}(f^k)}\exp(S_k\varphi(x)).$$

Apart from its capability as a definition when the expression is point dependent, it avoids referencing $X_{\mathtt{i}}$, which could be cumbersome.

Potentials Throughout we assumed that the potentials were twice differentiable with continuous derivative. However, it suffices for the maps to be $C^{1+\varepsilon}$. That is, the maps are differentiable, with Hölder continuous derivative.

Expanding maps Similarly, the mapping itself only has to be $C^{1+\varepsilon}$ on a suitable (e.g. convex, open domain). In some cases it may be easier to define the mappings in terms of (potentially overlapping) inverses F_i. The minimum assumptions to use the above results with minimal alteration are that the $F_i : \mathbb{R}^d \to \mathbb{R}^d$ are contracting, with (absolute) derivative uniformly bounded away from 0 and 1 on an open convex domain. In higher dimensions we have to assume that the derivative is a similarity, which also implies conformality (angle-preserving).

5.2 Dimension theory of cookie-cutters

In this section we will explore what we can learn about the dimension theory of attractors/repellers arising from cookie cutter systems. We will do this by trying to find a potential such that the associated Gibbs measure behaves like the Hausdorff measure on the repeller. That is, we want to find potential φ such that the Gibbs measure μ associated with that potential is equivalent to $\mathcal{H}^s|_E$, where E is the cookie cutter set and $s = \dim_H E$.

5.2.1 The upper bound

First note that the Hausdorff measure can be approximated from above through images of the attractor. That is,

$$\mathcal{H}^s_\delta(E) \le \sum_{\mathtt{i}\in I_k} \operatorname{diam}(X_\mathtt{i})^s$$

if k is large enough such that $\operatorname{diam}(X_\mathtt{i}) < \delta$ for all $\mathtt{i} \in I_k$. This gives

$$\mathcal{H}^s(E) \le \liminf_{k\to\infty} \sum_{\mathtt{i}\in I_k} |X_\mathtt{i}|^s. \tag{5.2}$$

Observe that $X_\mathtt{i}$ is comparable to $\frac{d}{dx}(F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_k}(x))$ and define $\phi(x) = -s\log|f'(x)|$. The pressure becomes

$$\begin{aligned}
P(-s\log|f'|) &= \lim_{k\to\infty} \frac{1}{k} \log \sum_{\mathtt{i}\in I_k} \exp\left(-\sum_{j=0}^{k-1} s\log|f'(f^j(x_\mathtt{i}))| \right) \\
&= \lim_{k\to\infty} \frac{1}{k} \log \sum_{\mathtt{i}\in I_k} |(f^k)'(x_\mathtt{i})|^{-s} \\
&= \lim_{k\to\infty} \frac{1}{k} \log \sum_{\mathtt{i}\in I_k} |X_\mathtt{i}|^s.
\end{aligned}$$

This looks similar to the Hausdorff measure bound in (5.2) and in fact,

$$\sum_{\mathtt{i}\in I_k} |X_\mathtt{i}|^s \approx \exp(kP(-s\log|f'(x)|)).$$

This shows, that we want the pressure to be zero for the associated Gibbs measure to relate directly to geometric sizes. Before we show that the s for which the pressure is zero also gives the Hausdorff dimension of the set, we need to establish a few intermediate results.

Lemma 5.2. *For $s \in \mathbb{R}$ and $\delta > 0$ we have*

$$-\delta m_2 \le P\left(-(s+\delta)\log|f'|\right) - P\left(-s\log|f'|\right) \le \delta m_1$$

where

$$0 < m_1 := \inf_{x\in X_1\cup X_2} \log|f'(x)| \le \sup_{x\in X_1\cup X_2} \log|f'(x)| =: m_2 < \infty$$

Proof. For $\delta > 0$,

$$\frac{1}{k}\log\left[\sum_{\mathtt{i}\in I_k} \exp\left(-\sum_{j=0}^{k-1} (s+\delta)\log|f'(f^j(x_\mathtt{i}))| \right) \right]$$

$$\leq \frac{1}{k}\log\left[\sum_{\mathtt{i}\in I_k}\exp\left(-\sum_{j=0}^{k-1} s\log|f'(f^j(x_{\mathtt{i}}))|\exp(-\delta k m_1)\right)\right]$$

$$\leq \frac{1}{k}\log\sum_{\mathtt{i}\in I_k}\exp\left(-\sum_{j=0}^{k-1} s\log|f'(f^j(x_{\mathtt{i}}))|\right) - \delta m_1.$$

Taking limits we obtain the upper bound. The lower bound proof is similar and omitted. □

Corollary 5.3. *$P_s = P(-s\log|f'|)$ is strictly decreasing and continuous, $P_s \to -\infty$ as $s \to \infty$ and $P_s \to \infty$ as $s \to -\infty$. Further, $P_0 = \log 2$ and hence there exists a unique $s > 0$ such that $P_s = 0$.*

Proof. The proof is left as an exercise. □

Theorem 5.4. *Let s be the unique solution to*

$$P(-s\log|f'|) = 0.$$

Then, $\dim_H E = s$ and $\mathcal{H}^s(E) \in (0,\infty)$. Further, $\mathcal{H}^s|_E$ is a Gibbs measure, i.e.

$$c^{-1}|X_{\mathtt{i}}|^s \leq \mathcal{H}^s(E\cap X_{\mathtt{i}}) \leq c|X_{\mathtt{i}}|^s$$

for all $\mathtt{i} \in I_k$ and $k \in \mathbb{N}$.

Proof. Letting s be the unique solution for zero pressure and taking the potential $\phi(x) = -s\log|f'(x)|$ we obtain a Gibbs measure μ satisfying

$$C^{-1} \leq \frac{\mu(X_{\mathtt{i}})}{\exp(S_k\phi(x))} \leq C.$$

By the chain rule,

$$C_1^{-1} \leq \frac{\mu(X_{\mathtt{i}})}{|(f^k)'(x)|^{-s}} \leq C_1.$$

and

$$C_2^{-1} \leq \frac{\mu(X_{\mathtt{i}})}{|X_{\mathtt{i}}|^s} \leq C_2.$$

Using the bounded distortion condition, every $B(x,r)$ with $x \in E$ and $r > 0$ contains a "not too small" cylinder set $X_{\mathtt{i}}$. That is, there exists $c > 0$ such that for all $x \in E, 0 < r < \mathrm{diam}(E)$ there exists $k \in \mathbb{N}$ and $\mathtt{i} \in I_k$ such that $|X_{\mathtt{i}}| > cr$ and $X_{\mathtt{i}} \subseteq B(x,r)\cap E$. Hence,

$$\mu(B(x,r)\cap E) \geq \mu(X_{\mathtt{i}}) \geq C_2^{-1}|X_{\mathtt{i}}|^s \geq \frac{c^s}{C_2}r^s.$$

Similarly one can show that $\mu(B(x,r)\cap E) \leq C_3 r^s$ for some $C_3 > 0$ [Exercise]. We conclude that there exists a measure μ and constant $C_4 > 0$ such that

$$C_4^{-1}r^s \leq \mu(B(x,r)\cap E) \leq C_4 r^s.$$

Using Proposition 3.5 we immediately get that

$$\mathcal{H}^s(X_{\mathtt{i}}) \approx \mu(X_{\mathtt{i}}) \approx |X_{\mathtt{i}}|^s$$

which proves the theorem. □

6 Quasi self-similar sets and their Hausdorff measure

Recall that we proved that cookie cutters have the property that every small ball centred on them contains a not too distorted copy of the entire set, which is not too small. We will make a property out of this condition and call sets that satisfy it quasi self-similar.

Definition 6.1. *Let E be a non-empty compact subset of $\mathbb{R}^d$. Let $D > 0$ and suppose that for every closed ball $B(x, r)$ with $x \in E$ and $0 < r < |E|$ there exists a mapping $g : E \to E \cap B(x, r)$ such that*

$$D^{-1}r|x - y| \le |g(x) - g(y)| \le Dr|x - y|$$

for all $x, y \in E$. We say that E is quasi-self-similar (QSS) with distortion $D > 0$.

6.1 Upper bounds on the Hausdorff measure

We can use the quasi self similarity to show that every such set must have finite Hausdorff measure. The proof uses the QSS condition and the maps g to create an "iterated function system" that generates a natural and uniform cover.

Theorem 6.2. *Let E be a QSS set with distortion $D > 0$. Then every r-packing with disjoint centred closed balls of radius $r < \min\{|E|, D\}$ has cardinality less than $D^s r^{-s}$, where $s = \dim_H E$.*

Further, $\mathcal{H}^s(E) \le 4^s D^s < \infty$.

Proof. We write $N_r(E)$ for the maximal cardinality of an r packing of E. Since E is bounded and $\mathbb{R}^d$ is doubling, E is totally bounded and so $N_r(E)$ is well-defined. Assume for a contradiction that

$$N_r(E) > D^s r^{-s}$$

for some $0 < r < \min\{|E|, D\}$. Since $r < D$ and so $(D/r)^s \ge 1$, there must exist $t > s$ such that $N_r(E) < D^t r^{-t}$. Thus there are disjoint balls $B_1, B_2, \ldots, B_N$ of radius r, centred in E, where we have written $N = N_r(F)$ to avoid unnecessary notation. By the QSS condition there exist $g_i : E \to E \cap B_i$ such that

$$|g_i(x) - g_i(y)| \ge D^{-1}r|x - y|. \tag{6.1}$$

Let $d = \min_{i \ne j} \operatorname{dist}(B_i, B_j) > 0$. Let $i_n, j_n \in \{1, \ldots, N\}$ and assume $\mathtt{i} = i_1 \ldots i_q$ and $\mathtt{j} = j_1 \ldots j_q$ agree only up to $(q - 1)$ for some $q \ge 1$. Then, by iterating the lower bound (6.1) $(q - 1)$ times gives

$$\begin{aligned} d(g_{i_1} \circ \cdots \circ g_{i_q}(E), g_{j_1} \circ \cdots \circ g_{j_q}(E)) &= d(g_{i_1} \circ \cdots \circ g_{i_q}(E), g_{i_1} \circ \cdots \circ g_{i_{q-1}} \circ g_{j_q}(E)) \\ &\ge (D^{-1}r)^{q-1} \operatorname{dist}(B_{i_q}, B_{j_q}) \ge d(r/D)^{q-1} \end{aligned}$$

Fix $z \in E$ arbitrarily. For $m \in \mathbb{N}$ we define μ_m to be the uniform discrete probability measure giving each

$$g_{i_1} \circ \cdots \circ g_{i_m}(z)$$

weight N^{-m} for all $i_n \in \{0, \ldots, N\}$. Our standard machinery, see e.g. Theorem 1.8 implies that $\mu_m \to \mu$ weakly, where μ is Borel. In particular, $\mu(g_{i_1} \circ \cdots \circ g_{i_q}(E)) = N^{-q}$.

Now, let $B(x, \rho)$ be a ball centred in E with diameter $2\rho < d$. Let k be the unique integer such that

$$(D^{-1}r)^{k+1}d \le 2\rho < (D^{-1}r)^k d.$$

The ball intersects at most one $g_{i_1} \circ \cdots \circ g_{i_k}(E)$ and so

$$\mu(B(x,\rho)) \leq N^{-k} < (D^{-1}r)^{kt} \leq \left(D^{-1}dr\right)^{-t}(2\rho)^t.$$

By the mass distribution principle we must have $\dim_H E \geq t > s$, which is a contradiction. This proves the first claim that $N_r(E) \leq D^s r^{-s}$.

The second claim can be proven by noting that doubling the radius in a maximal packing makes it a cover of the space. Hence, E can be covered by $N(r)$ many balls of radius $2r$ and diameter $4r$. Hence

$$\mathcal{H}^s_{4r}(E) \leq D^s r^{-s}(4r)^s = 4^s D^s$$

for all $r < D$. Hence $\mathcal{H}^s(E) \leq 4^s D^s$. □

Remark 6.3. *Note that we are really only using the lower bound of the quasi self similarity condition. However, the upper condition is necessary for many other properties of quasi self-similar sets.*

6.2 Geometrical Homogeneity

In this section we study the geometrical homogeneity of some sets in terms of the Hausdorff measure. We first look at self-similar sets.

6.2.1 Self-similar sets

Recall that a set E is called self-similar, if it is a non-empty compact set invariant under a finite number of contracting similarity maps, i.e. there exist similarity mappings $F_1, \ldots, F_N$ with

$$|F_i(x) - F_i(y)| = c_i|x-y|$$

for some $c_i \in (0,1)$ such that $E = \bigcup_{i=1}^N F_i(E)$. As it turns out self-similar sets are very homogeneous and have equal Hausdorff measure and content.

Proposition 6.4. *Let E be a self-similar set. Then $\mathcal{H}^s(E) = \mathcal{H}^s_\infty(E)$ and so $\mathcal{H}^s(A) = \mathcal{H}^s_\infty(A)$ for all $A \subseteq E$, where $s = \dim_H E$.*

Proof. First, $\mathcal{H}^s_\infty(E) < \infty$ as E is necessarily compact. We may also assume that $\mathcal{H}^s_\infty(E) > 0$ as the desired result holds trivially otherwise. Let $\varepsilon > 0$ and let $\{U_i\}_{i\in\Lambda}$ be a countable cover of closed balls such that

$$\mathcal{H}^s_\infty(E) \geq \sum_{i\in\Lambda} |U_i|^s - \varepsilon.$$

Now let B be the closed ball of diameter $|E|$ such that $E \subseteq B$. Let $\mathcal{V} = \{F_{\mathtt{i}}(B) : \mathtt{i} \in \{1,\ldots,N\}^*\}$ be the collection of all images of B. This is a closed Vitali cover and there exists $\mathcal{V}' \subset \mathcal{V}$ such that

$$\mathcal{H}^s(E) = \mathcal{H}^s\left(E \cap \bigcup_{\mathtt{i}:F_{\mathtt{i}}(B)\in\mathcal{V}'} F_{\mathtt{i}}(E)\right) = \sum_{\mathtt{i}:F_{\mathtt{i}}(B)\in\mathcal{V}'} \mathcal{H}^s\left(E \cap F_{\mathtt{i}}(E)\right) \leq \sum_{\mathtt{i}:F_{\mathtt{i}}(B)\in\mathcal{V}'} (c_{\mathtt{i}})^s\,\mathcal{H}^s(E)$$

by a variant[4] of the Vitali covering theorem. We conclude[5] that $\sum_{\mathtt{i}\in\mathcal{V}'}(c_{\mathtt{i}})^s \geq 1$. An "obvious" covering gives the reverse inequality.

[4] The variant is essentially the same as the Vitali covering theorem where the images $E \cap F_{\mathtt{i}}(B)$ replace the closed balls. We will not prove this result, but it may be attempted as a difficult exercise

[5] We are misusing notation here slightly. $\mathtt{i}$ is of course not an element of $\mathcal{V}'$, but it stands shorthand for "The $\mathtt{i}$ such that $F_{\mathtt{i}}(B) \in \mathcal{V}'$.

We now obtain

$$\begin{aligned}\mathcal{H}^s(E) &= \mathcal{H}^s\left(E \cap \bigcup_{\mathtt{i}\in\mathcal{V}'} F_{\mathtt{i}}(E)\right) \le \mathcal{H}^s\left(\bigcup_{\mathtt{i}\in\mathcal{V}'}\bigcup_{j\in\Lambda} F_{\mathtt{i}}(U_j)\right)\\ &= \sum_{\mathtt{i}\in\mathcal{V}'}(c_{\mathtt{i}})^s\,\mathcal{H}^s\left(\bigcup_{j\in\Lambda} U_j\right)\left(\sum_{\mathtt{i}\in\mathcal{V}'}(c_{\mathtt{i}})^s\right)\sum_{j\in\Lambda}|U_j|^s\\ &\le \mathcal{H}^s_\infty(E)+\varepsilon\end{aligned}$$

and as $\varepsilon > 0$ was arbitrary our first claim follows. The second claim follows immediately from Theorem 2.10. □

6.2.2 Quasi self similar sets

We now move onto quasi self similar sets. First, recall that Hausdorff measure and content for quasi self-similar sets cannot always be equal. The upper half-circle can be checked to be quasi self similar. However its content is 2 but its measure is π. This suggests that the Hausdorff content and measure may be related by a multiplicative constant, and indeed one can prove that these notions are related.

Lemma 6.5. *Let $E \subseteq \mathbb{R}^d$ be quasi self similar. Let $s = \dim_H F$ and write $N_r(E)$ for the maximal cardinality of disjoint r-packings of E. Then,*

$$2^{-s}\,\mathcal{H}^s_\infty(E) r^s \le N_r(E) \le D^s r^{-s}.$$

Proof. The proof is left as an exercise. □

Exercise 6.1. *Prove Lemma 6.5.*

Theorem 6.6. *Let E be a quasi self-similar set. Then there exists $C > 0$ such that for all $x \in E$ and $r > 0$,*

$$\mathcal{H}^s(F \cap B(x,r)) \le C r^s$$

and

$$\mathcal{H}^s_\infty(F\cap A) \le \mathcal{H}^s(F\cap A) \le C\,\mathcal{H}^s_\infty(F\cap A)$$

for all $A \subseteq \mathbb{R}^d$.

Proof. We may assume $\mathcal{H}^s(E) > 0$ as the proof is trivial otherwise. But then $\mathcal{H}^s_\infty(E) > 0$ and we write

$$C = 2\cdot 2^{3s} D^{3s}\,\mathcal{H}^s_\infty(E)^{-1}.$$

Assume for a contradiction that there exist $x_0 \in E$ and $r_0 > 0$ such that $\mathcal{H}^s(E\cap B(x_0,r_0)) > Cr_0^s$. Fix $n \in \mathbb{N}$ and let $\mathcal{B}_n$ be a 2^{-n}-packing of E. We have

$$2^{-s}\,\mathcal{H}^s_\infty(E)2^{ns} \le \#\mathcal{B}_n \le D^s 2^{ns}. \tag{6.2}$$

For $B \in \mathcal{B}_n$ let g_B be the map guaranteed by the QSS condition. Then, for every $B \in \mathcal{B}_n$,

$$\begin{aligned}\mathcal{H}^s(g_B(E\cap B(x_0,r_0))) &\ge D^{-s}2^{-ns}\,\mathcal{H}^s(E\cap B(x_0,r_0))\\ &> CD^{-s}2^{-ns}r_0^s = 2\cdot 2^{3s-ns}D^{2s}\,\mathcal{H}^s_\infty(E)^{-1}r_0^s.\end{aligned} \tag{6.3}$$

Since $\operatorname{diam}(g_B(E \cap B(x_0, r_0))) \leq D2^{-n}2r_0 =: \delta_n$, we get

$$\mathcal{H}^s_{\delta_n}(g_B(E \cap B(x_0, r_0))) \leq D^s 2^{-ns} 2^s r_0^s \tag{6.4}$$

for all $B \in \mathcal{B}_n$. Combining (6.2) and (6.3) gives

$$\sum_{B \in \mathcal{B}_n} \mathcal{H}^s(g_B(E \cap B(x_0, r_0))) \geq 2 \cdot 2^{2s} D^{2s} r_0^s$$

and combining (6.2) with (6.4) gives

$$\sum_{B \in \mathcal{B}_n} \mathcal{H}^s_{\delta_n}(g_B(E \cap B(x_0, r_0))) \leq 2^{2s} D^{2s} r_0^s.$$

In turn, we get

$$\begin{aligned}
\mathcal{H}^s(E) &= \mathcal{H}^s\left(F \setminus \bigcup_{B \in \mathcal{B}_n} g_B(E \cap B(x_0, r_0))\right) + \sum_{B \in \mathcal{B}_n} \mathcal{H}^s(g_B(E \cap B(x_0, r_0))) \\
&\geq \mathcal{H}^s\left(F \setminus \bigcup_{B \in \mathcal{B}_n} g_B(E \cap B(x_0, r_0))\right) + 2 \cdot 2^{2s} D^{2s} r_0^s
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}^s_{\delta_n} &\leq \mathcal{H}^s_{\delta_n}\left(F \setminus \bigcup_{B \in \mathcal{B}_n} g_B(E \cap B(x_0, r_0))\right) + \sum_{B \in \mathcal{B}_n} \mathcal{H}^s_{\delta_n}(g_B(E \cap B(x_0, r_0))) \\
&\leq \mathcal{H}^s\left(F \setminus \bigcup_{B \in \mathcal{B}_n} g_B(E \cap B(x_0, r_0))\right) + 2^{2s} D^{2s} r_0^s.
\end{aligned}$$

So,

$$\mathcal{H}^s(E) - \mathcal{H}^s_{\delta_n}(E) \geq 2 \cdot 2^{2s} D^{2s} r_0^s - 2^{2s} D^{2s} r_0^s = 2^{2s} D^{2s} r_0^s > 0$$

which is independent of δ_n. Hence taking $n \to \infty$ ad $\delta_n \to 0$ we get a contradiction. This proves our first claim. The second claim is left as an exercise. $\square$

Exercise 6.2. *Prove that the Hausdorff content (up to a constant) is an upper bound to the Hausdorff dimension for any subset of $\mathbb{R}^d$.*

A standard way to refer to global and local homogeneity is through the Ahlfors-David regularity.

Definition 6.7. *Let $E \subseteq \mathbb{R}^d$. The set E is said to be Ahlfors-David s-regular if there exists a Radon measure μ with support E satisfying*

$$C^{-1} r^s \leq \mu(B(x, r)) \leq C r^s$$

for some uniform $C > 0$.

We can use the result above to show that all quasi self similar sets are Ahlfors-David s-regular, precisely when they have dimension s and positive s-dimensional Hausdorff measure.

Corollary 6.8. *Let E be a quasi self similar set with Hausdorff dimension s. Then, E is Ahlfors-David s-regular if and only if $\mathcal{H}^s(E) > 0$.*

Proof. We start by proving that $\mathcal{H}^s(E) > 0$ implies Ahlfors-David regularity. Consider $\mu = \mathcal{H}^s|_E > 0$, then

$$\mu(B(x,r)) = \mathcal{H}^s(E \cap B(x,r)) \le Cr^s$$

by the last Theorem. Further, since E is quasi-self-similar there exists $g_B E \to E \cap B(x,r)$ that maps with distortion $D > 0$. Then,

$$\mathcal{H}^s(E \cap B(x,r)) \ge \mathcal{H}^s(g_B(E)) \ge D^{-s} r^s\, \mathcal{H}^s(E).$$

Since $\mathcal{H}^s(E) > 0$ is a constant and $\mathcal{H}^s|_E$ is Radon (see exercise below), Ahlfors regularity holds with constant $C' = \max\{C, D^s\, \mathcal{H}^s(E)^{-1}\}$.

To prove the converse, assume that E is Ahlfors-David s-regular. Let U_i be a countable δ-cover of E. Write B_i for a ball of radius $4|U_i|$ that contains U_i and is centred in U_i. Then,

$$\mu(E) \le \sum_i \mu(U_i) \le \sum_i \mu(B_i) \le C \sum_i |2U_i|^s$$

and so $\mathcal{H}^s_\delta(E) \ge 2^{-s} C^{-1} \mu(E) > 0$ for all $\delta > 0$. Hence $\mathcal{H}^s(E) \ge \mu(E)/(2^s C) > 0$, which completes the proof. ☐

Exercise 6.3. *Assume E is an Ahlfors-David s-regular set. Show that* $\dim_H E = s$.

Exercise 6.4. *Show that $\mathcal{H}^s|_E$ is a Radon measure whenever E is a QSS set. What are the requirements for the (restricted) Hausdorff measure to be Radon?*

7 Structure of s-sets and Ahlfors-David regular sets

Recall the Lebesgue density theorem.

Theorem 7.1. *Lebesgue density theorem Let $E \subseteq \mathbb{R}^d$ be a Borel set. Then, for $\mathcal{L}^d$-almost all $x \in \mathbb{R}^d$,*

$$\lim_{r \to 0} \frac{\mathcal{L}^d(E \cap B(x,r))}{\mathcal{L}^d(B(x,r))} = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Given that the Hausdorff measure can be considered an extension of the Lebesgue measure to non-integer dimension, we may want to study densities with respect to the Hausdorff measure[6].

Definition 7.2. *Let $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$. We write*

$$\overline{D}^s(E,x) = \limsup_{r \to 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{\mathcal{H}^s_\infty(B(x,r))} = \limsup_{r \to 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{2^s r^s}$$

and

$$\underline{D}^s(E,x) = \liminf_{r \to 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{2^s r^s}$$

for the upper and lower density of E at x, respectively. If both limits coincide, we write $D^s(E,x)$ for the density of E at x.

[6]Technically, we have to use the Hausdorff content in the denominator. This is due to the fact that the s-dimensional Hausdorff measure of a d-dimensional ball is infinite for $s < d$, whereas the content gives us the right “intuition” for volume

We restrict our attention at first to so-called s-sets. Sets that satisfy a minimum of "regularity" with respect to the Hausdorff measure.

Definition 7.3. *Let $E \subseteq \mathbb{R}^d$. We say that E is an s-set if $0 < \mathcal{H}^s(E) < \infty$.*

Note that this immediately implies that the Hausdorff dimension of E is s.

Exercise 7.1. *Let $E \subset \mathbb{R}^d$ be bounded. Show that any Ahlfors-David s-regular set is also an s-set*

Definition 7.4. *A point $x \in \mathbb{R}^d$ is called regular if $D(E,x) = 1$. It is called irregular otherwise.*

Similarly, a set $E \subseteq \mathbb{R}^d$ is called regular if $\mathcal{H}^s$-almost every point is regular. If $\mathcal{H}^s$-almost every point is irregular, E is called irregular.

Remark 7.5. *While a point is irregular if it is not regular, this is not true for sets. There could, a priori, be sets which have both regular and irregular points with positive Hausdorff measure.*

Proposition 7.6. *Let $E \subseteq \mathbb{R}^d$ be an s-set. Then,*

1. $\underline{D}^s(E,x) = \overline{D}^s(E,x) = 0$ *for $\mathcal{H}^s$-almost every $x \notin E$,*
2. $2^{-s} \leq \overline{D}^s(E,x) \leq 1$ *for $\mathcal{H}^s$-almost every $x \in F$.*

Proof. (1) is easy if E is closed, since E^c is an open set and so any small enough ball will not intersect E. The general case is quite difficult to prove and is left out.

(2) follows from the definition of density (upper bound) and Proposition 3.5, see exercise below. ☐

Exercise 7.2. *Finish the proof of statement (2) above.*

As it turns out, the Lebesgue density theorem does not hold for the Hausdorff measure. In fact, it fails "spectacularly" as the following result shows.

Theorem 7.7. *Let $E \subseteq \mathbb{R}^d$ be an s-set. Then E is irregular unless $s \in \mathbb{N}$.*

Unfortunately, a full proof is out of the scope of this course. We will however give the proof for a reduced version of the theorem that allows for a nice geometrical argument.

Theorem 7.8. *Let $E \subseteq \mathbb{R}^2$ be an s-set for $s \in (0,1)$. Then E is irregular.*

Proof. Assume by the way of a contradiction that E is not irregular. Thus, there exists a subset of $E'_R \subseteq E$ with positive Hausdorff measure containing only regular points. We may assume that E'_R is Borel. Using Proposition 7.6, we see that $D(E,x) \geq 2^{-s}$ for (almost) all $x \in E'_R$.

Recall Egorov's theorem, which states that given a sequence f_n of measurable functions on some measure space (X, Σ, μ), and given a measurable subset $A \subseteq X$ of finite μ measure such that f_n converges μ almost surely to some f. Then f_n converges uniformly on a subset $A' \subseteq A$ with measure $\mu(A') > \mu(A) - \varepsilon$ for all $\varepsilon > 0$.

Let $f_n(x) = \mathcal{H}^s(E \cap B(x, r_n))/(2r_n)^s$ for $r_n = \alpha^n$ for some $2^{s-1} < \alpha^s < 1$. We can apply Egorov's theorem to show that for some Borel set $E_R \subseteq E'_R$ with positive Hausdorff measure and some $r_0 > 0$,

$$\mathcal{H}^s(E \cap B(x,r)) > \alpha^s D^s(E,x)(2r)^s \geq 2^{s-1}2^{-s}(2r)^s = \tfrac{1}{2}(2r)^s$$

for all $x \in E_R$ and $r < r_0$. Let $y \in E_R$ be a limit point[7]. Now, let $\delta > 0$ be arbitrary and for every $r > 0$ consider the annulus $A(r, \delta) = B(y, r(1+\delta)) \setminus B(y, r(1-\delta))$. Clearly,

$$\begin{aligned}\mathcal{H}^s(E \cap A(r,\delta))/(2r)^s &= (2r)^{-s}\left(\mathcal{H}^s(E \cap B(y, r(1+\delta))) - \mathcal{H}^s(E \cap B(y, r(1-\delta)))\right)\\ &\to D^s(E,y)((1+\delta)^s - (1-\delta)^s)\end{aligned}$$

for $r \to 0$. Since y was a limit point, we may take a sequence of $x_n \to y$, writing $r_n = |x_n - y|$. Then, $B(x, r\delta/2)$ is contained in the annulus and

$$\tfrac{1}{2}(r_n\delta)^s < \mathcal{H}^s(E \cap B(x_n, r\delta/2)) \le \mathcal{H}^s(E \cap A(r_n, \delta)).$$

Note that the Taylor expansion for x^s around 1 is $1 + s(x-1) + O((x-1)^2)$. Therefore,

$$(1+\delta)^s - (1-\delta)^s = 1 + s\delta + O(\delta^2) - (1 - s\delta + O(\delta^2)) = 2s\delta + O(\delta^2)$$

and

$$\begin{aligned}(2r_n)^{-s}\,\mathcal{H}^s(E \cap A(r_n,\delta)) &> \tfrac{1}{2}\left(\frac{r_n\delta}{2r_n}\right)^s = 2^{-(1+s)}\delta^s\\ \Rightarrow \mathcal{H}^s(E \cap A(r_n,\delta)) &> 2^{-1}\delta^s r_n^s\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Then, for small enough r_n,

$$\mathcal{H}^s(E \cap A(r_n,\delta)) \le (1+\varepsilon)(2r_n)^s D(E,y)(2s\delta + O(\delta^2))$$

and

$$2^{-1}\delta^s r_n^s < (1+\varepsilon)2^s r_n^s D(E,y)(2s\delta + O(\delta^2)) \le (1+\varepsilon)2^s r_n^s(2s\delta + O(\delta^2)).$$

This gives

$$2^{-1-s} < (1+\varepsilon)(2s\delta^{1-s} + \delta^{-s}O(\delta^2))$$

which clearly does not hold for small $\delta > 0$. This is our required contradiction. □

7.1 Structure of 1-sets and rectifiability

We saw that s-sets for non-integer values must be irregular. We will now investigate sets where $s = 1$. Our first result shows that we can decompose any such set into a regular and irregular part.

Theorem 7.9 (Decomposition theorem)**.** *Let E be a 1-set. Then the set of regular points in E forms a regular set and the remainder is an irregular set.*

Proof. This follows from the observation that the density of a point $x \in F \subset E$ is almost surely the same with respect to either E or F, assuming F is Borel. Formally,

$$\frac{\mathcal{H}^s(E \cap B(x,r))}{(2r)^s} = \frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s} + \frac{\mathcal{H}^s((E \setminus F) \cap B(x,r))}{(2r)^s}.$$

and since

$$\frac{\mathcal{H}^s((E \setminus F) \cap B(x,r))}{(2r)^s} \to 0$$

for $\mathcal{H}^s$-almost all $x \in E$ we get

$$\overline{D}^s(F,x) = \overline{D}^s(E,x) \quad \text{and} \quad \underline{D}^s(F,x) = \underline{D}^s(E,x).$$

The conclusion of the theorem follows immediately. □

[7] Why must there be limit points in E_R?

Note that either of the two sets may be empty. A smooth curve, for example is a 1-set with no irregular part (of positive Hausdorff measure). The irregular set may even be empty, such as for the unit circle.

On the opposing end, the "Cantor dust" that is the invariant set of the four contractions $f_1(x) = x/4$, $f_2(x) = x/4 + (0, 3/4)$, $f_3(x) = x/4 + (3/4, 0)$, and $f_4(x) = x/4 + (3/4, 3/4)$ is an s-set that only contains irregular points.

This is a typical scenario and we will be able to decompose sets into a "line like" and "dust like" part.

Definition 7.10. *A* Jordan curve *is the image of a continuous injection* $\phi : [0, 1] \to \mathcal{C} \subset \mathbb{R}^2$. *The length of* $\mathcal{C}$ *is*

$$\mathcal{L}(\mathcal{C}) = \sup \sum_{i=1}^{m} |x_i - x_{i-1}|$$

where the supremum is taken of all finite tuples of points $x_0, \ldots, x_m$ *such that* $\phi^{-1}(x_i) < \phi^{-1}(x_{i+1})$.

If $\mathcal{L}(\mathcal{C})$ *is positive and finite, we say that* $\mathcal{C}$ *is a* rectifiable curve.

Note that any Jordan curve defined in this way is not self-intersecting and has two distinct ends.

Lemma 7.11. *If* $\mathcal{C}$ *is a rectifiable curve, then* $\mathcal{H}^1(\mathcal{C}) = \mathcal{L}(\mathcal{C})$

Proof. Let $x, y \in \mathcal{C}$ and write $L(x, y) \subset \mathbb{R}^2$ for the line through x and y and $\overline{L}(x, y) \subset L(x, y)$ for the line segment starting at x and ending at y. Note that the natural orthogonal projection $\pi_{L(x,y)}$ is a Lipschitz map which, in particular, does not increase distances. Thus, $\mathcal{H}^s(A) \geq \mathcal{H}^s(\pi_{L(x,y)} A)$ for all $A \subset \mathbb{R}^2$. In particular, let $\mathcal{C}(x, y)$ be the closed section of $\mathcal{C}$ starting at x and ending at y. Then,

$$\mathcal{H}^1(\mathcal{C}(x, y)) \geq \mathcal{H}^1(\pi_{L(x,y)}(\mathcal{C}(x, y))) \geq \mathcal{H}^1(\overline{L}(x, y)) = |x - y|.$$

Therefore, for any finite partition,

$$\sum_{i=1}^{m} |x_i - x_{i-1}| \leq \sum_{i=1}^{m} \mathcal{H}^1(\mathcal{C}(x_i, x_{i-1})) \leq \mathcal{H}^1(\mathcal{C}).$$

Taking suprema over all partitions gives $\mathcal{H}^1(\mathcal{C}) \geq \mathcal{L}(\mathcal{C})$.

To prove the opposite inequality, observe that $\phi' : [0, \mathcal{L}(\mathcal{C})] \to \mathcal{C}$ that maps $t \in [0, \mathcal{L}(\mathcal{C})]$ to the point in $\mathcal{C}$ that is t away from $\phi(0)$ is a rectifiable curve that has been reparametrised. Since then $|\phi'(t) - \phi'(s)| \leq |t - s|$ for all $s, t \in [0, \mathcal{L}(\mathcal{C})]$ the mapping ϕ' is Lipschitz and

$$\mathcal{H}^1(\mathcal{C}) \leq \mathcal{H}^1([0, \mathcal{L}(\mathcal{C})]) = \mathcal{L}(\mathcal{C}). \qquad \square$$

It is straightforward to show that every rectifiable curve is a regular 1-set.

Proposition 7.12. *Let* $\mathcal{C}$ *be a rectifiable curve. Then* $\mathcal{C}$ *is a regular* 1*-set.*

Sketch of proof. Take $y \in \mathcal{C}$ not an endpoint. For $r > 0$ sufficiently small, and since $\mathcal{C}$ is continuous and not self-intersecting, there must be two distinct points $x, z \in \mathcal{C}$ at distance r from y. Then,

$$\frac{\mathcal{H}^1(\mathcal{C} \cap B(x, r))}{(2r)^1} \geq \frac{\mathcal{H}^1(\mathcal{C}(x, y) \cap B(x, r))}{2r} + \frac{\mathcal{H}^1(\mathcal{C}(y, z) \cap B(x, r))}{2r} \geq \frac{r}{2r} + \frac{r}{2r} = 1$$

giving $\underline{D}^1(\mathcal{C}, y) \geq 1$ from which the result follows. $\square$

We define *curve-like* sets to capture this regularity.

Definition 7.13. *A* 1*-set* E *is curve-like if it is contained in a countable union of rectifiable curves.*[8]

As one would expect, adding countably many curves does not change regularity.

Proposition 7.14. *Let* E *be a curve-like set. Then* E *is a regular.*

Proof. Index the countable rectifiable curves $\mathcal{C}_i$ by $i \in \mathbb{N}$. Given $x \in E$, let i_x be the least integer such that $x \in D_{i_x} \cap E$. Then,

$$\underline{D}^1(E, x) \geq \underline{D}^1(E \cap \mathcal{C}_{i_x}, x) = \underline{D}^1(\mathcal{C}_{i_x}, x) = 1$$

for $\mathcal{H}^1$-almost every $x \in E \cap \mathcal{C}_{i_x}$. But since the collection is countable, this holds for $\mathcal{H}^1$-almost every $x \in E$. □

A complimentary definition is being curve-free, also known as unrectifiability.

Definition 7.15. *Let* E *be a* 1*-set. If* $\mathcal{H}^1(E \cap \mathcal{C}) = 0$ *for every rectifiable curve* $\mathcal{C}$*, we say that* E *is curve-free or unrectifiable.*

It is easy to see that any irregular 1-set E is unrectifiable. Consider any rectifiable curve $\mathcal{C}$. Then $E \cap \mathcal{C}$ is a subset of a regular and irregular 1-set. As such it must have zero measure.

Proposition 7.16. *Let* E *be an irregular* 1*-set. Then* E *is unrectifiable.*

We can complete characterise regular and irregular sets.

Theorem 7.17.

1. *A* 1*-set in* $\mathbb{R}^2$ *is irregular if and only if it is unrectifiable.*
2. *A* 1*-set in* $\mathbb{R}^2$ *is regular if and only if it is the union of a curve-like set and a set of zero* $\mathcal{H}^1$*-measure (i.e. it is rectifiable).*

Proof. (1) An irregular set is curve free by Proposition 7.16. We will omit a proof of the converse, which follows from the stronger fact that any curve-free 1-set in $\mathbb{R}^2$ has lower density less than 3/4 for $\mathcal{H}^1$-almost every point.

(2) By Proposition 7.14 any curve-like set is regular. This is unaltered if we include a zero measure set.

Proving the converse is slightly more involved. Assume E is regular. Then any Borel subset $F \subseteq E$ has density $D^1(F, 1) = 1$ for almost all $x \in F$. By the above mentioned fact on densities, the set F cannot be curve free and there exist rectifiable curves that intersect F with positive length. Using induction we first define $\mathcal{C}_1$ to be any rectifiable curve such that

$$\mathcal{H}^1(E \cap \mathcal{C}_1) \geq \tfrac{1}{2} \sup \left\{ \mathcal{H}^1(E \cap \mathcal{C}) : \mathcal{C} \text{ is a rectifiable curve} \right\}.$$

Having defined $\mathcal{C}_1, \ldots, \mathcal{C}_k$, we define $\mathcal{C}_{k+1}$ by considering $E_k = E \setminus \bigcup_{i=1}^k \mathcal{C}_i$ which is a regular set of positive measure. We let $\mathcal{C}_{k+1}$ be any rectifiable curve such that

$$\mathcal{H}^1(E_k \cap \mathcal{C}_{k+1}) \geq \tfrac{1}{2} \sup \left\{ \mathcal{H}^1(E_k \cap \mathcal{C}) : \mathcal{C} \text{ is a rectifiable curve} \right\}$$

[8] A similar notion is a countably rectifiable set, which is any set that is contained in a countable connection of rectifiable curves up to a set of zero measure.

If the process terminates, we must have exhausted the measure and our claim holds. Otherwise,

$$\sum_{k=1}^{\infty} \mathcal{H}^1(E_k \cap \mathcal{C}_{k+1}) \leq \mathcal{H}^1(E) < \infty. \tag{7.1}$$

Now assume for a contradiction that $\mathcal{H}^1(E \backslash \bigcup_{i=1}^{\infty} \mathcal{C}_i) > 0$. Since the set is regular there exists a rectifiable curve $\mathcal{C}$ that intersects it with positive length, say L. But $\mathcal{H}^1(E_k \cap \mathcal{C}_{k+1}) < L/2$ for some k as the sum in (7.1) converges. But then $\mathcal{H}^1(E_k \cap \mathcal{C}_{k+1}) \geq L$ and $\mathcal{C}$ would have been picked over $\mathcal{C}_{k+1}$, a contradiction. □

Remark 7.18. *It is noteworthy that this full classification relates two notions that seem at first very disparate: the local densities and the geometric intuition of rectifiability.*

In fact, the above shows that regular sets are essentially subsets of unions of rectifiable curves, whereas irregular sets contain no part of a rectifiable curve.

Remark 7.19. *This also shows that regular* 1*-sets may be connected, but irregular* 1 *are totally disconnected as no two points can lie in the same connected component.*

Remark 7.20. *Using stronger assumptions such as Ahlfors David regularity, one can show that the notion of unrectifiability is equivalent to that of zero analytic capacity.*

8 Projections of sets and Frostman's Lemma

Recall the potent mass distribution principle (slightly paraphrased), Lemma 2.8.

Lemma 8.1. *Let $E \subseteq \mathbb{R}^d$ be Borel and let $s > 0$. If there exists a Borel measure μ with* $\operatorname{supp} \mu \subseteq E$ *and $\mu(E) > 0$ such that $\mu(B(x,r)) \leq r^s$ for all $x \in \mathbb{R}^d$ and $r > 0$, then $\mathcal{H}^s(E) > 0$.*

Recall also that the proof does not require μ to be a Borel measure at all, just an outer measure. The reason for the wording is that the converse also holds.

Lemma 8.2. *Let $E \subseteq \mathbb{R}^d$ be Borel and let $s > 0$. The following are equivalent:*

- *There exists a Borel measure μ with* $\operatorname{supp} \mu \subseteq E$ *and $\mu(E) > 0$ such that $\mu(B(x,r)) \leq r^s$ for all $x \in \mathbb{R}^d$ and $r > 0$.*
- *$\mathcal{H}^s(E) > 0$.*

While one direction is usually called the mass distribution principle, the converse is called Frostman's Lemma after Otto Frostman, who discovered the equivalency in his PhD thesis for closed sets. While this makes a (very challenging) exercise, the full equivalency for Borel sets is long an tough to prove and we will omit either proof.

8.1 Potential theoretic methods

One can use Frostman's Lemma and the equivalency above to prove a corollary that expresses the measure properties in *energy integrals*, which provide surprisingly strong methodology.

Let μ be a finite and positive Borel measure on $E \subseteq \mathbb{R}^d$. Then, the s energy of E with respect to μ is the double integral

$$I_\mu^s(E) = \iint_{E \times E} \frac{d\mu(x) d\mu(y)}{|x-y|^s}.$$

For integer values, these energy integrals may be familar to anyone who studied some physics. For $s = 1$ this corresponds to the potential energy of particles distributed according to μ with a force that satisfies an inverse square law, such as gravity.

It turns out that this notion is very useful due to its connection to the Hausdorff measure.

Corollary 8.3. *1. Let E be a Borel set and μ be a positive and finite Borel measure supported on E. Then,*

$$I^s_\mu(E) < \infty \quad \Rightarrow \quad \mathcal{H}^s(E) = \infty.$$

2. Let E be a Borel set with $\mathcal{H}^s(E) > 0$. Then, for all $0 < t < s$ there exists a positive and finite Borel measure μ supported on E such that $I^s_\mu(E) < \infty$.

8.2 Projections in the plane

Studying projections to distinguish and classify their objects is an essential part of many fields. In many contexts we are only given a projected image of an object, say the 2-dimensional projection of a 3-dimensional object on our retinas, or screens. Yet, we are still able to reconstruct, with great accuracy, the 3-dimensional properties of the objects we are observing. Other effects in nature also arise from projections of higher dimensional objects. The structure of quasi-crystals and their unusual lattice structures can be easily understood by higher dimensional lattices that are projected into three dimensional space.

In this last section we will try to understand the dimensional properties of such projections and study the “size” of objects when projected into lower dimensional spaces. For ease, we will later work in two dimensions only, but the methods extend without issue into higher dimensional spaces.

8.2.1 Orthogonal projections and Marstrand’s projection theorem

We saw that the orthogonal projections $\pi_\theta : \mathbb{R}^2 \to L_\theta$, where L_θ is the line through the origin at angle θ, are Lipschitz maps already. This holds in general, any orthogonal projections $\pi : \mathbb{R}^n \to \mathbb{R}^m$ cannot increase distances. Hence, by the definition of the Hausdorff measure, $\mathcal{H}^s(\pi E) \le \mathcal{H}^s(E)$ for any $E \subseteq \mathbb{R}^d$. This also implies, for any set $E \in \mathbb{R}^n$ and orthogonal projection $\pi : \mathbb{R}^n \to \mathbb{R}^m$,

$$\dim_H \pi E \le \min\{m, \dim_H E\},$$

that is, dimensions cannot increase and the dimension is bounded by the ambient space dimension.

Thinking about this inequality in the plane and projecting onto lines we see that this is certainly not sharp. Letting E be a straight line segment, there exists an angle such that the projection is a single point. However, for all other angles the projection is a line segment of reduced length. We may guess that our upper bound is correct in “most” cases, and this is guaranteed by the famous Marstrand projection theorem.

Theorem 8.4. *Let $E \subseteq \mathbb{R}^2$ be Borel.*

1. *If $\dim_H E \le 1$, then $\dim_H \pi_\theta(E) = \dim_H E$ for almost every $\theta \in [0, \pi)$.*
2. *If $\dim_H E > 1$, then $\mathcal{L}^1(\pi E) > 0$ for almost every $\theta \in [0, \pi)$.*

Proof. Let $s < \dim_H E \le 1$. Then, by Corollary 8.3 there exists a positive and finite Borel measure μ for which

$$\iint_{E\times E} \frac{d\mu(x)d\mu(y)}{|x-y|^s} < \infty.$$

Let $\mu_\theta = (\pi_\theta)_*(\mu)$ be the projected measure onto L_θ, that is

$$\mu_\theta(A) = \mu\left\{x \in \mathbb{R}^2 : \pi(x) \in A\right\}$$

for all Borel $A \in L_\theta$. Equivalently, we may express the measure as an integral

$$\int_{-\infty}^{\infty} f(t) d\mu_\theta(t) = \int_E f(x \cdot \vec{\theta}) d\mu(x)$$

where $\vec{\theta}$ is the unit vector in direction θ and $\cdot$ is the usual dot product.

Our goal is to show that the energy of the projected measure $I^s_{\mu_\theta}(\pi E)$ is finite. This will imply a lower bound on the dimension by Corollary 8.3. However, it is impossible to do this for all directions and we use a trick to make it manageable. We will integrate the energy with respect to θ and show that this is finite.

$$\begin{aligned}
\int_0^\pi I^s_{\mu_\theta}(\pi E) &= \int_0^\pi \left[\int_0^\infty \int_0^\infty \frac{d\mu_\theta(u) d\mu_\theta(w)}{|u-w|^s}\right] d\theta \\
&= \int_0^\pi \left[\iint_{E\times E} \frac{d\mu(x)d\mu(y)}{|x\cdot\vec{\theta} - y\cdot\vec{\theta}|^s}\right] d\theta \\
&= \iint_{E\times E} \left(\int_0^\pi \frac{d\theta}{\vec{v}(x-y)\cdot\vec{\theta}}\right) \frac{d\mu(x)d\mu(y)}{|x-y|^s} \qquad \text{(Using Fubini)} \qquad (8.1)
\end{aligned}$$

where $\vec{v}(x-y)$ is the unit vector in the direction of $x-y$.

The innermost integral can be written as

$$\int_0^\pi \frac{d\theta}{\vec{v}(x-y)} = \int_0^\pi \frac{d\theta}{|\cos(\phi-\theta)|^s},$$

where θ is the angle of $\vec{v}(x-y)$. But since we are only considering the absolute value of the cosine, we are integrating over its whole period. In particular, this means that ϕ is only a phase shift and the value of the integral is independent of x and y. One can use the small angle approximation to show that $\cos(\phi-\theta) \approx \theta + c$ for θ close to the value that makes the cosine zero. This is also the only part that provides an issue with determining the integral. However, since $s < 1$, the integral must be bounded using basic calculus. We write c_s for the value of the integral, then (8.1) is bounded by

$$\leq c_s \iint_{E\times E} \frac{d\mu(x)d\mu(y)}{|x-y|^s} = c_s I^s_\mu(E) < \infty.$$

Thus $I^s_{\mu_\theta}(\pi E)$ is bounded for almost every θ and $\mathcal{H}^s(\pi_\theta E) = \infty$ and $\dim_H \pi E \geq s$. The conclusion follows by taking s arbitrarily close to $\dim_H E$.

We will not prove the statement about positive Lebesgue measure here as it is slightly beyond the scope of the course. However, it follows from similar ideas. □

Remark 8.5. *The theorem above was proven in this way by Marstrand in the plane using a combinatorial approach. The idea to use potential theoretic methods came from Kaufman, with whose methods it is straightforward to extend the proof to all orthogonal projection in (finite dimensional) Euclidean space.*

The generalisation to higher dimensions is straightforward, but requires a little more definitions. The set of all orthogonal projections $G(n,m)$ from $\mathbb{R}^n$ into $\mathbb{R}^m$ is called the Grassmannian manifold. It supports a natural Haar measure ν, with which we can chose "typical" projections.

Theorem 8.6 (Marstrand projection theorem (higher dimensional))**.** *Let $E \subseteq \mathbb{R}^n$. Then, for all $1 \leq m \leq n$ and ν-almost every $\pi \in G(n,m)$ we have*

$$\dim_H \pi E = \min\{m, \dim_H E\}.$$

Further, if $\dim_H E > m$, then $\mathcal{L}^m(\pi E) > 0$ for almost every $\pi \in G(n,m)$.

8.3 Projections and 1-sets

We end with a few results on projections of 1-sets.

Proposition 8.7. *Let $E \subseteq \mathbb{R}^2$ be an irregular 1 set. Then $\pi_\theta E$ has zero Lebesgue measure for almost every direction θ.*

Proof. The proof is intricate and omitted. □

Proposition 8.8. *Let $E \subseteq \mathbb{R}^2$ be a regular 1-set. Then $\mathcal{L}^1(\pi E) > 0$ for all but at most one exceptional direction.*

Proof. (Heuristics) Since regular 1-sets are rectifiable it must be contained in (multiple) rectifiable curves ith positive length. The only possibility for πE to have zero measure is if those curves are lines that align. □

And using our earlier decomposition and classification theorem we obtain

Corollary 8.9. *A 1-set is irregular if and only if it has zero $\mathcal{L}^1$ measure in two directions.*

References

[1] P. Mattila. *Geometry of sets and measures in Euclidean spaces, Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics, **44**. Cambridge University Press, Cambridge, 1995.