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Geometric Measure Theory
PhD course, HT26

Course Outline[1]: Geometric Measure Theory, HT26

The purpose of this PhD course is to study aspects of geometric measure theory from the viewpoint of dimension theory, dynamical and random constructions, as well as to study regularity and projection problems.

The main objects of study are Hausdorff measure and content, and we start off by looking at dimension theory to classify local and global regularity of metric spaces. We will look at sets exhibiting bounded distortion that arise from dynamical systems and their invariant measures, and use potential-theoretic methods in Kaufman's proof of Marstrand's projection theorem. The final part of the course will move towards current research topics through Orponen's projection theorem for Assouad dimension, the Kakeya problem, and Bourgain's discretised projection theorem. Wang and Zahl's resolution of the Kakeya conjecture in $\mathbb{R}^3$ was among the achievements recognised when Hong Wang was awarded the 2026 Fields Medal.

Midway through the course, Adam Śpiewak will give a three-lecture minicourse on embedding theory, reconstruction of dynamical systems, and multifractal analysis, which will form part of this course. Further, there will be a colloquium talk by Pablo Shmerkin in October that will likely touch upon several topics covered in the course.

Apart from the geometric and dynamical aspects, there is a continuous probabilistic thread throughout the course, as many concrete results can only be stated for typical realisations. These usually have a good probabilistic interpretation and will lead to powerful techniques.

Format and References: The course consists of 20 lectures loosely following the lecture plan below. The plan may be adjusted to the background and interests of the participants. Apart from the material in the lectures, lecture notes will be provided[2]. However, the main classical reference is Mattila [6] and for the later parts we will study (parts of): Orponen [8]; the resolution of the Kakeya problem in $\mathbb{R}^3$ by Wang and Zahl [12] and its streamlined proof by Guth, Wang, and Zahl [5]; and the streamlined version of Bourgain's theorem [1] by O'Regan, Shmerkin, and Wang [7].

Prerequisites: The course is intended for PhD students and advanced Master's students in mathematics. We assume some familiarity with graduate real analysis, including basic measure theory and integration as well as basic metric-space topology. Previous knowledge of geometric measure theory, probabilistic methods, dynamical systems, or additive combinatorics is not required.

Participation and resources: You do not need to inform me about participation, but I maintain an email list to send information about course changes and supplementary material. Please let me know if you want to be included by sending me a short email to: sascha.troscheit@math.uu.se.

All material such as the lecture notes, problems, and course information will be available on my webpage `http://www.troscheit.eu/GMT2026.html`, which can also be found through the "teaching tab".

Examination: Three problem sheets will be distributed during the course, each worth 100 points. The grading is pass/fail. To pass, one needs at least 20 points in each assignment and 120 points overall. Solutions are to be submitted handwritten (scanned) or in LaTeX and will be confirmed by a final oral examination.

[1] Please be aware that the document is subject to change as the course progresses. Last updated: August 23, 2026

[2] Please note that the current version available on my webpage is from a course I gave in Vienna some years ago. It will be updated throughout this semester with the new material.

Schedule and contents

All lectures take place in the mathematics seminar room Å/64119. The schedule is as follows. See also the TimeEdit Schedule.

Lecture	Date and time	Contents
L1	7 Sep, 13:15–15:00	Hausdorff content and outer measure.
L2	9 Sep, 15:15–17:00	Hausdorff measure, dimension, and Frostman lemma.
L3	14 Sep, 13:15–15:00	Covering theorems, densities, and regular sets.
L4	16 Sep, 10:15–12:00	Box-counting and packing dimensions.
L5	29 Sep, 13:15–15:00	Assouad dimension and local extremal scaling.
L6	30 Sep, 15:15–17:00	Iterated function systems and quasi-self-similarity.
L7	6 Oct, 13:15–15:00	Cookie-cutter sets and bounded distortion.
	7 Oct, 15:00	*Colloquium talk by Pablo Shmerkin.*
L8	13 Oct, 13:15–15:00	Pressure and the dimension of cookie-cutter sets.
L9	21 Oct, 15:15–17:00	Guest lecture I: embedding theory.
L10	27 Oct, 13:15–15:00	Guest lecture II: reconstructing dynamics from observations.
L11	28 Oct, 10:15–12:00	Guest lecture III: embeddings in fractal geometry.
L12	3 Nov, 13:15–15:00	Energies, capacities, and dimension.
L13	4 Nov, 10:15–12:00	Kaufman's proof of Marstrand's projection theorem.
L14	10 Nov, 13:15–15:00	Orponen's Assouad projection theorem I.
L15	11 Nov, 10:15–12:00	Orponen's Assouad projection theorem II.
L16	24 Nov, 13:15–15:00	The Kakeya problem: sets, tubes, and dimension.
L17	25 Nov, 10:15–12:00	Multiscale Kakeya and the projection connection.
L18	1 Dec, 13:15–15:00	Discretisation and the additive-combinatorial toolkit.
L19	2 Dec, 10:15–12:00	From projection to polynomial expansion.
L20	8 Dec, 13:15–15:00	Bourgain's projection theorem and outlook.

Detailed lecture plan

I. Measures and dimensions (Lectures 1–5)

Lecture 1: Hausdorff content and outer measure Metric outer measures, Hausdorff content $\mathcal{H}^s_\infty$, and the approximating quantities $\mathcal{H}^s_\delta$. Basic monotonicity and scaling properties, measurability, and some Cantor sets.

Lecture 2: Hausdorff measure, dimension, and Frostman lemma Critical exponents, countable stability, and the mass distribution principle. A compact version of Frostman's lemma.

Lecture 3: Covering theorems, densities, and regular sets The $5r$-covering lemma, Vitali covering, density estimates, and Ahlfors regular measures. The geometry of s-sets and rectifiability.

Lecture 4: Box-counting and packing dimensions Covering and packing numbers, lower and upper box-counting dimensions, packing premeasure, and packing dimension. The modified box-counting characterisation of packing dimension and the comparison inequalities.

Lecture 5: Assouad dimension and local extremal scaling The two-scale covering definition of Assouad dimension, bi-Lipschitz invariance, and weak tangents. Relation to doubling and measure analogues.

II. Dynamically defined sets (Lectures 6–8)

Lecture 6: Iterated function systems and quasi-self-similarity The Hutchinson operator, coding space, separation conditions, and self-similar measures. Bernoulli measures on coding space. Frostman measures and the Ahlfors regularity dichotomy for quasi-self-similar sets.

Lecture 7: Cookie-cutter sets and bounded distortion Expanding Markov maps of an interval, their repellers, and symbolic coding. Bounded distortion and natural covers.

Lecture 8: Pressure and the dimension of cookie-cutter sets Partition sums, subadditivity, and topological pressure for the geometric potential. Bowen's formula: the Hausdorff dimension of the repeller is the unique solution of $P(-s \log |T'|) = 0$.

III. Embeddings, potential theory, and projections (Lectures 9–13)

Lecture 9: Guest lecture I: embedding theory Introduction and historical development of embedding theory in differential geometry and topology. The focus then shifts to typical linear maps and orthogonal projections in geometric measure theory, where typicality is formulated using a natural measure on the parameter space. Mané's embedding theorem and a measure-theoretic version are proved. Time permitting, the lecture compares Hausdorff, box-counting, and Assouad dimension assumptions with the regularity that one can demand of an embedding.

Lecture 10: Guest lecture II: reconstructing dynamics from observations Time-delay coordinates in time-series analysis and the reconstruction and prediction of a dynamical system from observed data. The lecture develops Takens-type embedding theorems, gives proofs of representative versions, and introduces recent probabilistic approaches based on random observables and high-probability reconstruction.

Lecture 11: Guest lecture III: embeddings in fractal geometry Natural projections of conformal iterated function systems, the Feng–Hu dimension formula, and the relation between embeddings and overlaps. Applications to local dimensions and multifractal analysis.

Lecture 12: Energies, capacities, and dimension Riesz kernels, s-energy, and the layer-cake representation of energy. The energy form of Frostman's lemma and its probabilistic interpretation.

Lecture 13: Kaufman's proof of Marstrand's projection theorem Orthogonal projections of sets and measures, projected energies, and averaging over directions. Kaufman's integral estimate giving $\dim_{\mathrm{H}} \pi_e(F) = \min\{\dim_{\mathrm{H}} F, 1\}$ for almost every direction e when $F \subset \mathbb{R}^2$. Positive-Lebesgue-measure conclusions and exceptional directions.

IV. Assouad projections and Kakeya sets (Lectures 14–17)

Lecture 14: Orponen's Assouad projection theorem I Orponen's theorem [8] on the Assouad dimension of projections. Proof through weak tangents, Frostman measures on exceptional directions, and the extraction of a quasi-regular blow-up.

Lecture 15: Orponen's Assouad projection theorem II Dyadic tubes encode small projections of the regularised blow-up. We refine the tube families to controlled branching, locate product-like structure, and formulate the additive inverse theorem that supplies the contradiction.

Lecture 16: The Kakeya problem: sets, tubes, and dimension Besicovitch constructions and the Kakeya conjecture in Hausdorff and Minkowski forms. The L^2 incidence bound for δ-tubes. A first look at the planar theorem, the Wang–Zahl Assouad-dimension result [11], and their resolution of the conjecture in $\mathbb{R}^3$ [12].

Lecture 17: Multiscale Kakeya and the projection connection Direction-separated tube families, multiplicity, hairbrush ideas, and sticky versus non-sticky behaviour. Following Guth–Wang–Zahl [5], we go through the broad aspects of the proof in $\mathbb{R}^3$.

V. Bourgain's discretised projection theorem (Lectures 18–20)

Lecture 18: Discretisation and the additive-combinatorial toolkit (δ, s, C)-sets, non-concentration, and covering estimates. The Ruzsa triangle inequality, Plünnecke–Ruzsa theory, Balog–Szemerédi–Gowers, and Kaufman's projection estimate.

Lecture 19: From projection to polynomial expansion The expansion argument of O'Regan–Shmerkin–Wang using Marstrand's theorem, Blichfeldt's principle, and coefficient elimination. Growth of $A + xA$ for non-concentrated sets.

Lecture 20: Bourgain's projection theorem and outlook Bourgain's robust discretised projection theorem via product structure, additive structure, and polynomial expansion. Applications to Furstenberg sets, Kakeya, distance problems, and dynamics.

Indicative references

[1] J. Bourgain, *The discretized sum-product and projection theorems*, Journal d'Analyse Mathématique 112 (2010), 193–236.

[2] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, third edition, Wiley, 2014.

[3] D.-J. Feng and H. Hu, *Dimension theory of iterated function systems*, Communications on Pure and Applied Mathematics 62 (2009), 1435–1500.

[4] J. M. Fraser, *Assouad Dimension and Fractal Geometry*, Cambridge University Press, 2021.

[5] L. Guth, H. Wang, and J. Zahl, *A streamlined proof of the Kakeya set conjecture in* $\mathbb{R}^3$, arXiv:2601.14411, 2026.

[6] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[7] W. O'Regan, P. Shmerkin, and H. Wang, *Simple proofs of discretised projection theorems*, arXiv:2511.21656, 2025.

[8] T. Orponen, *On the Assouad dimension of projections*, Proceedings of the London Mathematical Society 122 (2021), 317–351.

[9] T. Sauer, J. A. Yorke, and M. Casdagli, *Embedology*, Journal of Statistical Physics 65 (1991), 579–616.

[10] F. Takens, *Detecting strange attractors in turbulence*, Lecture Notes in Mathematics 898 (1981), 366–381.

[11] H. Wang and J. Zahl, *The Assouad dimension of Kakeya sets in* $\mathbb{R}^3$, Inventiones Mathematicae 241 (2025), 153–206.

[12] H. Wang and J. Zahl, *Volume estimates for unions of convex sets, and the Kakeya set conjecture in three dimensions*, arXiv:2502.17655, 2025.