

The total amount of points available on this sheet is 100 points, divided equally into four themes. You may use the results proved through Lecture 5. State clearly which result you use and check its hypotheses. Unless a different metric is specified, all sets carry the Euclidean metric.

Submit handwritten (scanned) or typeset solutions by email to sascha.troscheit@math.uu.se or leave them in my pigeonhole by the due date/time.

Question 1: Hausdorff measure and Frostman measures.

- a) Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be non-empty compact sets. Prove that

$$\dim_{\mathrm{H}}(A \times B) \geq \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B.$$

Construct non-empty compact sets $A, B \subseteq [0, 1]$ such that

$$\dim_{\mathrm{H}} A = \dim_{\mathrm{H}} B = 0, \quad \dim_{\mathrm{H}}(A \times B) = 1.$$

(7p)

- b) Let C be the middle-third Cantor set, $s = \log 2 / \log 3$, and let E_n be the set of left endpoints of the 2^n level- n construction intervals of C . For $\tau > 1$ put

$$W_\tau = \{x \in C : |x - e| < 3^{-\tau n} \text{ for some } e \in E_n, \text{ for infinitely many } n\}.$$

Show that $\dim_{\mathrm{H}} W_\tau \leq s/\tau$ and that $\dim_{\mathrm{P}} W_\tau = \underline{\dim}_{\mathrm{B}} W_\tau = \overline{\dim}_{\mathrm{B}} W_\tau = s$. (8p)

- c) Show that $\dim_{\mathrm{H}} W_\tau = s/\tau$. (10p)

Turn the page.

Question 2: Coverings, density, and regularity.

- a) For $p > 0$, let

$$F_p = ([0, 1] \times \{0\}) \cup \bigcup_{n=2}^{\infty} (\{1/n\} \times [0, n^{-p}]).$$

Show that F_p is countably 1-rectifiable. Determine for which p it has finite length and for which p it is Ahlfors 1-regular. Compute

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^1(F_p \cap B(0, r))}{2r}$$

in $[0, \infty]$. For $p > 1$, explain why the behaviour at the origin is compatible with density one almost everywhere. (10p)

- b) Let $E \subseteq \mathbb{R}^d$ be non-empty and bounded. Suppose there are $0 < \theta < 1$ and $r_0 > 0$ such that every ball $B(x, r)$ with $x \in E$ and $0 < r < r_0$ contains a ball of radius θr disjoint from E . Prove that $\dim_{\text{B}} E < d$.

Hint. Find a fixed integer k such that every sufficiently small dyadic cube has a descendant k levels below it disjoint from E . (5p)

- c) Let $0 < s < 1$. Construct a compact set $E \subseteq [0, 1]$ such that

$$0 < \mathcal{H}^s(E) < \infty, \quad \underline{D}^s(E, x) = 0 \quad \text{for every } x \in E.$$

(10p)

Question 3: Box and packing dimensions.

- a) **Parallel sets.** Let $C \subseteq [0, 1]$ be the middle-third Cantor set, put $s = \log 2 / \log 3$, and for $r > 0$ define

$$C_r = \{x \in \mathbb{R} : \text{dist}(x, C) < r\}.$$

Find the exact value of $\mathcal{L}^1(C_r)$ when $3^{-(n+1)} \leq 2r \leq 3^{-n}$, where $n \geq 0$ is an integer. Determine

$$\liminf_{r \downarrow 0} r^{s-1} \mathcal{L}^1(C_r) \quad \text{and} \quad \limsup_{r \downarrow 0} r^{s-1} \mathcal{L}^1(C_r),$$

and prove that these two constants are distinct. (4p)

- b) **Hölder maps.** Let $A \subseteq \mathbb{R}^m$ and let $f: A \rightarrow \mathbb{R}^d$ satisfy $|f(x) - f(y)| \leq L|x - y|^\alpha$ for all $x, y \in A$, where $0 < \alpha \leq 1$ and $L > 0$.

- (i) If A is non-empty and bounded, prove that

$$\underline{\dim}_B f(A) \leq \alpha^{-1} \underline{\dim}_B A, \quad \overline{\dim}_B f(A) \leq \alpha^{-1} \overline{\dim}_B A.$$

- (ii) Prove that $\dim_P f(A) \leq \alpha^{-1} \dim_P A$ for arbitrary A .

- (iii) Sharpness: let C be the middle-third Cantor set and let μ be the probability measure on C giving each level- n construction interval mass 2^{-n} . Set $s = \log 2 / \log 3$ and $\phi(x) = \mu([0, x])$ for $x \in C$.

Show that ϕ is s -Hölder and maps C onto $[0, 1]$. Deduce that all three dimension bounds are sharp when $\alpha = s$.

(5p)

- c) Let $\varphi(x) = \text{dist}(x, \mathbb{Z})$, let $0 < \alpha \leq \beta < 1$, and let $(a_k)_{k \geq 0}$ be sequence such that

$$2^{-\beta} \leq \frac{a_{k+1}}{a_k} \leq 2^{-\alpha} \quad (k \geq 0).$$

Put

$$f(x) = \sum_{k=0}^{\infty} a_k \varphi(2^k x),$$

and let $\Gamma = \{(x, f(x)) : x \in [0, 1]\} \subseteq \mathbb{R}^2$ be the graph of f over $[0, 1]$. Show that for every dyadic interval $I = [p2^{-n}, (p+1)2^{-n}] \subseteq [0, 1]$

$$\frac{1}{2}a_n \leq \sup_I f - \inf_I f \leq c a_n,$$

where c depends only on α and β .

(7p)

- d) for the function in (c), prove that

$$\underline{\dim}_B \Gamma = 2 - \limsup_{n \rightarrow \infty} \frac{\log_2(1/a_n)}{n}, \quad \overline{\dim}_B \Gamma = \dim_P \Gamma = 2 - \liminf_{n \rightarrow \infty} \frac{\log_2(1/a_n)}{n}.$$

For $\alpha < \beta$, choose (a_k) such that $\dim_H \Gamma \leq 2 - \beta < 2 - \alpha = \dim_P \Gamma$.

(9p)

Question 4: Assouad dimension, lower dimension, and regularity.

- a) For $E \subseteq \mathbb{R}^d$ with at least two points, the *lower dimension* of E is

$$\dim_L E = \sup \left\{ s \geq 0 : \begin{array}{l} \text{there exists } c > 0 \text{ such that, for every } x \in E \\ \text{and } 0 < r < R \leq \text{diam } E, \quad N_r(E \cap B(x, R)) \geq c(R/r)^s \end{array} \right\},$$

and $\dim_L E = 0$ if E has at most one point. Prove that $\dim_L E \leq \dim_H E$ for every closed set $E \subseteq \mathbb{R}^d$. Show that this inequality fails for $E = \mathbb{Q} \cap [0, 1]$, and that lower dimension is not monotone, even on compact sets.

(9p)

- b) Show that a compact Ahlfors s -regular set $E \subseteq \mathbb{R}^d$ satisfies $\dim_{\text{L}} E = s$. Conversely, construct a compact set $E \subseteq \mathbb{R}$ with $0 < \dim_{\text{L}} E = \dim_{\text{A}} E$ that is not Ahlfors regular for any exponent. (10p)
- c) Below is an AI generated proof that shows that the Assouad dimension is Lipschitz stable (which it is not). Find the flaw in the reasoning. Under what additional hypotheses would the statement hold? (6p)

Claim. Let $E \subseteq \mathbb{R}^m$ be non-empty and bounded, and let $f: E \rightarrow \mathbb{R}^d$ be L -Lipschitz, with $L > 0$. Then $\dim_{\text{A}} f(E) \leq \dim_{\text{A}} E$.

Proof. Let $s > \dim_{\text{A}} E$, and choose $C \geq 1$ with $N_a(E \cap B(x, b)) \leq C(b/a)^s$ for all $x \in E$ and $0 < a < b$. Fix $y \in f(E)$ and $0 < r < R$, and put $A = \{x \in E : f(x) \in B(y, R)\}$, so that $f(E) \cap B(y, R) = f(A)$.

Step 1. Let $x \in E$. Cover $E \cap B(x, R/L)$ by at most $C(2R/r)^s$ open balls of radius $r/(2L)$. Discard the balls that miss E , and replace each remaining ball by the ball of radius r/L about a point of E in it. The new balls still cover $E \cap B(x, R/L)$, and f maps the part of E in each of them into a ball of radius r . Hence

$$N_r(f(E \cap B(x, R/L))) \leq 2^s C(R/r)^s.$$

Step 2. Since A is bounded, it contains a maximal finite set $\{x_1, \dots, x_M\}$ of points pairwise at distance at least R/L . By maximality, the balls $B(x_i, R/L)$ cover A . The balls $B(x_i, R/(2L))$ are pairwise disjoint and centred in A , so they form a packing of A . Since f is L -Lipschitz, it maps $E \cap B(x_i, R/(2L))$ into $B(f(x_i), R/2)$. Thus f carries this packing of A to the packing $\{B(f(x_i), R/2)\}_{i=1}^M$ of $f(A)$, and $M \leq M_{R/2}(f(A))$. Every ball $B(f(x_i), R/2)$ lies in $B(y, 3R/2)$, and comparing Lebesgue measure gives $M_{R/2}(f(A)) \leq 3^d$. Hence $M \leq 3^d$.

Step 3. By Steps 1 and 2,

$$N_r(f(E) \cap B(y, R)) \leq \sum_{i=1}^M N_r(f(E \cap B(x_i, R/L))) \leq 3^d 2^s C(R/r)^s.$$

This bound is uniform in $y \in f(E)$ and $0 < r < R$, so $\dim_{\text{A}} f(E) \leq s$. Letting $s \downarrow \dim_{\text{A}} E$ proves the claim.