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ASSIGNMENT
PhD course, HT26
Problem Sheet 2
Geometric Measure Theory
Due: 2 December 2026, 23:59

The sheet has four questions worth 25 points each, for a total of 100 points. You may use the results from the lecture notes and Adam Śpiewak's minicourse notes. State clearly which result you use and check its hypotheses.

Question 1: A Gibbs measure for a quadratic potential.

Let C be the middle-third Cantor set, with $f_i(x) = (x+2i)/3$ for $i \in \{0,1\}$ and $T(f_i(x)) = x$ for $x \in [0,1]$. Use the word notation $C_w = f_w(C)$ and put $\psi(x) = x^2$. For $s \in \mathbb{R}$, let $P(s)$ be the pressure of $s\psi$ and let μ_s be a Gibbs measure obtained by the weak-limit construction in Lecture 8, with reference point 0. You may use the lecture's variation, partition-sum, and Gibbs estimates, and assume that $P \in C^1(\mathbb{R})$. All logarithms are natural.

a) Prove that, for $s, t \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \int_C e^{tS_n\psi(x)}\, d\mu_s(x) = P(s+t) - P(s).$$

(3p)

b) For every $s \in \mathbb{R}$, prove that μ_s is exact dimensional, i.e. that

$$\lim_{r\to 0} \frac{\log \mu_s(B(x,r))}{\log r} = \dim_{\mathrm{H}} \mu_s = \frac{P(s) - sP'(s)}{\log 3}$$

for μ_s-almost every x. *(9p)*

c) Put $d = \log 2/\log 3$. Identify μ_0. For $s \neq 0$, show that μ_s is neither comparable to $\mathcal{H}^d|_C$ nor a Bernoulli self-similar measure for f_0, f_1. *Hint.* Compare the fixed points $0, 1$ and the period-two orbit $\{1/4, 3/4\}$. *(6p)*

d) Let $h(x) = (x + x^2)/2$, $E = h(C)$, $F = h \circ T \circ h^{-1}$, and $\widetilde{\psi} = \psi \circ h^{-1}$. Show that F is a cookie-cutter map with repeller E and that its pressure satisfies, for $s, t \in \mathbb{R}$,

$$P_F(s\widetilde{\psi} - t \log |F'|) = P(s) - t \log 3.$$

Find $t = t(s)$ for which $s\widetilde{\psi} - t \log |F'|$ has zero pressure, and prove that $h_*\mu_s$ is Gibbs for this potential. Identify the potential at $s = 0$. *(7p)*

Turn the page.

Question 2: Hausdorff dimension and linear embeddings.

Take compact sets $X, Y \subseteq [0,1]$ with

$$\dim_{\mathrm{H}} X = \dim_{\mathrm{H}} Y = 0, \qquad X + Y \supseteq [0,1].$$

You may use the complementary binary-digit construction from Question 1(a) of Sheet 1, including the fact that its two sets have sum $[0,1]$. There is no need to repeat that construction here.

a) Let $W = (1 - X) \cup Y$. Show that W is a compact subset of $[0,1]$ with

$$\dim_{\mathrm{H}} W = 0, \qquad W - W \supseteq [-1,1].$$

(4p)

b) Put

$$K = (W \times \{0\}) \cup (W \times \{1\}) \cup (\{0\} \times W) \cup (\{1\} \times W) \subseteq \mathbb{R}^2.$$

Show that K is compact with $\dim_{\mathrm{H}} K = 0$, and that for every $u \in \mathbb{R}^2 \setminus \{0\}$ there are distinct $x, y \in K$ with $x - y \in \mathbb{R}u$. Deduce that no linear map $L\colon \mathbb{R}^2 \to \mathbb{R}$ is injective on K. Explain why this rules out replacing upper box dimension by Hausdorff dimension in the hypothesis of Mañé's embedding theorem. *(10p)*

c) Compute $\dim_{\mathrm{H}}(K - K)$ and $\overline{\dim}_{\mathrm{B}} K$. *(11p)*

Question 3: Injectivity and regularity of the inverse.

a) Let μ be a Borel probability measure on $\mathbb{R}^2$ with

$$\dim_{\mathrm{H}} \mu = 0, \qquad \operatorname{supp} \mu = [0,1]^2.$$

Prove that there is no linear map $L\colon \mathbb{R}^2 \to \mathbb{R}$ and no Borel set A with $\mu(A) = 1$ such that $L|_A$ is injective with a continuous inverse, despite almost every L being injective on a set of full measure. *(8p)*

b)

For $m \geq 1$ let $N_m = 2^{m^2}$ and

$$U_m = \left\{ \left(\cos \frac{\pi j}{N_m}, \sin \frac{\pi j}{N_m} \right) : j = 0, 1, \ldots, N_m - 1 \right\},$$

$$X = \{0\} \cup \bigcup_{m \geq 1} 2^{-m} U_m.$$

Let μ be a probability measure on X with $\mu(\{x\}) > 0$ for every $x \in X$. Then almost every linear map $L\colon \mathbb{R}^2 \to \mathbb{R}$ is injective on a set of full μ-measure, with a continuous inverse. Is there a range of $\alpha \in (0,1]$ for which almost every L has a pointwise α-Hölder inverse at μ-almost every x? *(10p)*

c) Calculate $\overline{\dim}_{\mathrm{B}} X$. For the lower bound, use Part (b) and the regularity theorem for measures, [1, Theorem 2.2(B)(ii)]. *(7p)*

Question 4: Delay coordinates and predictability.

Let $S^1 = \{e^{it} : t \in \mathbb{R}\} \subseteq \mathbb{R}^2$, and for $\theta \in [0, 2\pi)$ let $T_\theta(e^{it}) = e^{i(t+\theta)}$. For an observable $h\colon S^1 \to \mathbb{R}$ and $k \in \mathbb{N}$, put

$$\varphi_{h,k}(x) = \big(h(x), h(T_\theta x), \dots, h(T_\theta^{k-1} x)\big).$$

Recall that h is *deterministically k-predictable* if, for all $x_1, x_2 \in S^1$,

$$\varphi_{h,k}(x_1) = \varphi_{h,k}(x_2) \quad \Longrightarrow \quad \varphi_{h,k}(T_\theta x_1) = \varphi_{h,k}(T_\theta x_2).$$

a) Show that $\varphi_{h,1}$ is injective for no continuous $h\colon S^1 \to \mathbb{R}$. *(5p)*

For the remaining parts, let $h_0(e^{it}) = \cos t$.

b) Determine all $\theta \in [0, 2\pi)$ for which h_0 is deterministically 1-predictable. *(6p)*

c) Determine all $\theta \in [0, 2\pi)$ for which $\varphi_{h_0,2}$ is injective on S^1. *(9p)*

d) For $\theta = \pi$, is there k for which $\varphi_{h_0,k}$ is injective? *(5p)*

References

[1] A. Śpiewak, *Embedding theory in geometric measure theory*, lecture notes for a minicourse at Uppsala University, version of 27 October 2026. `https://www.troscheit.eu/embedding_minicourse_notes.pdf`.