Embedding theory in geometric measure theory

Part 1: Basics of embedding theory

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The embedding problem

Goal: embed a set X (or a measure μ) into a Euclidean space $\mathbb{R}^k$.

Embedding: at least an injective map $\phi\colon X \to \mathbb{R}^k$; in the best case an isomorphism onto the image in the considered category (homeomorphism, bi-Lipschitz map, diffeomorphism, ...).

For a measure μ: injectivity only on a set of full μ-measure.

> *What information about the size/complexity of X (or μ) is needed to guarantee existence (or abundance) its embeddings into $\mathbb{R}^k$?*

The answer is usually given in terms of the **dimension** of X (or μ).

Existence versus genericity

Given a class $\mathcal{C}$ of maps $X \to \mathbb{R}^k$ (continuous, smooth, Lipschitz, linear, delay-coordinate maps, natural projections of IFS, ...):

- **Existence:** is there an embedding $\phi\colon X \to \mathbb{R}^k$ in the class $\mathcal{C}$?
- **Genericity:** is a *typical* map $\phi \in \mathcal{C}$ an embedding?

"Typical": dense G_δ / Lebesgue-a.e. parameter of a finite-dimensional family / prevalent.

In this course: genericity, in the setting of geometric measure theory.

(Q1) How large must k be, in terms of the dimension of X (or μ), for a typical $\phi \in \mathcal{C}$ to be injective (resp. μ-a.s. injective)?

(Q2) How regular is the inverse ϕ^{-1}?

(Q3) Which notion of dimension governs the answers to (Q1)–(Q2)?

Plan of the course

- **Part 1:** history and Mañé's embedding theorem (1st lecture)
- **Part 2:** quality (regularity) of the embeddings (2nd lecture)
- **Part 3:** applications to reconstruction of dynamics from data (2nd-3rd lecture)
- **Part 4:** applications to (multi)fractal geometry (3rd lecture)

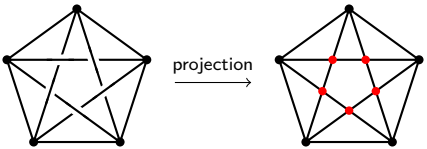
Topological embeddings

Theorem (Menger 1926, Nöbeling 1931)

Let X be a compact metric space with $\dim_{\mathrm{T}} X \leq n$. Then embeddings $X \to \mathbb{R}^{2n+1}$ form a dense G_δ subset of $C(X, \mathbb{R}^{2n+1})$.

Example

The n-skeleton of the $(2n+2)$-simplex does not embed into $\mathbb{R}^{2n}$. For $n = 1$: K_5 is not planar.



$K_5 \subset \mathbb{R}^3$ image in $\mathbb{R}^2$

Smooth embeddings

Theorem (Weak Whitney embedding theorem, 1936)

Let M be a compact C^r manifold of dimension n, $r \geq 1$. Then C^r embeddings $M \to \mathbb{R}^{2n+1}$ form an open and dense subset of $C^r(M, \mathbb{R}^{2n+1})$.

- Every abstract manifold is a submanifold of a Euclidean space.

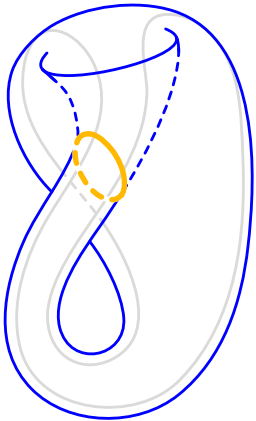
Theorem (Strong Whitney embedding theorem, 1944)

Every C^r manifold of dimension $n \geq 1$, $r \geq 1$, admits a C^r embedding into $\mathbb{R}^{2n}$.

- Weak: genericity. Strong: existence only (transverse double points persist under perturbations).
- $2n$ is optimal in general: $\mathbb{RP}^n \not\hookrightarrow \mathbb{R}^{2n-1}$ if n is a power of 2.

Example: the Klein bottle

- The Klein bottle K is a compact surface ($n = 2$).
- K embeds into $\mathbb{R}^4 = \mathbb{R}^{2n}$ (strong Whitney theorem).
- K does not embed into $\mathbb{R}^3$: every continuous map $K \to \mathbb{R}^3$ has self-intersections.



Dimensions of sets

Let X be a metric space and $N(X, r)$ the minimal number of r-balls needed to cover X.

Definition

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \inf \Big\{ \sum_i \operatorname{diam}(U_i)^s : X \subset \bigcup_i U_i,\ \operatorname{diam}(U_i) \le \delta \Big\},$$

$$\dim_{\mathrm{H}} X = \inf\{s : \mathcal{H}^s(X) = 0\},$$

$$\underline{\dim}_{\mathrm{B}} X = \liminf_{r \to 0} \frac{\log N(X, r)}{-\log r}, \qquad \overline{\dim}_{\mathrm{B}} X = \limsup_{r \to 0} \frac{\log N(X, r)}{-\log r},$$

$$\overline{\dim}_{\mathrm{B}} X = \inf \Big\{ s \ge 0 :\ \exists\, M > 0 \ \ \forall\, 0 < r < 1 \quad N(X, r) \le M \Big(\frac{1}{r}\Big)^s \Big\},$$

$$\dim_{\mathrm{A}} X = \inf \Big\{ s \ge 0 :\ \exists\, M > 0 \ \ \forall\, x \in X,\ 0 < r < R \quad N(X \cap B(x, R), r) \le M \Big(\frac{R}{r}\Big)^s \Big\}.$$

Basic properties

Proposition

- $\dim_{\mathrm{T}} X \leq \dim_{\mathrm{H}} X \leq \underline{\dim}_{\mathrm{B}} X \leq \overline{\dim}_{\mathrm{B}} X \leq \dim_{\mathrm{A}} X$ *(all can be strict).*
- f *α-Hölder* $\Rightarrow \dim_{\mathrm{H}} f(X) \leq \dim_{\mathrm{H}} X/\alpha$, $\overline{\dim}_{\mathrm{B}} f(X) \leq \overline{\dim}_{\mathrm{B}} X/\alpha$
- $\dim_{\mathrm{A}}$ *may increase under Lipschitz maps.*
- $\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y \leq \dim_{\mathrm{H}}(X \times Y)$
- $\overline{\dim}_{\mathrm{B}}(X \times Y) \leq \overline{\dim}_{\mathrm{B}} X + \overline{\dim}_{\mathrm{B}} Y$
- $\dim_{\mathrm{A}}(X \times Y) \leq \dim_{\mathrm{A}} X + \dim_{\mathrm{A}} Y$.

Dimensions of measures

Definition

Let μ be a finite Borel measure on $\mathbb{R}^N$. Local dimensions:

$$\underline{d}(\mu, x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}, \qquad \overline{d}(\mu, x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

μ is *exact dimensional* if $\underline{d}(\mu, x) = \overline{d}(\mu, x) = \text{const}$ for μ-a.e. x. Hausdorff dimensions:

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} A : \mu(\mathbb{R}^N \setminus A) = 0\}, \qquad \underline{\dim}_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} A : \mu(A) > 0\}.$$

Proposition

$\dim_{\mathrm{H}} \mu = \operatorname*{ess\,sup}_{x \sim \mu} \underline{d}(\mu, x)$ *and* $\underline{\dim}_{\mathrm{H}} \mu = \operatorname*{ess\,inf}_{x \sim \mu} \underline{d}(\mu, x)$.

Dimension of a measure vs. dimension of its support

Example

1. $\mu = \sum_n 2^{-n}\delta_{q_n}$, $\{q_n\}$ dense in $[0,1]^N$: $\dim_{\mathrm{H}} \mu = 0$, but $\dim_{\mathrm{H}} \operatorname{supp} \mu = N$.
2. μ_p – distribution of $\sum_{i\geq 1} \xi_i 2^{-i}$, ξ_i i.i.d. with $\mathbb{P}(\xi_i = 1) = p$, $\mathbb{P}(\xi_i = 0) = 1 - p$: $\operatorname{supp} \mu_p = [0,1]$, but

$$\dim_{\mathrm{H}} \mu_p = \frac{-p\log p - (1-p)\log(1-p)}{\log 2} < 1 \qquad \text{for } p \neq \tfrac{1}{2}.$$

Orthogonal projections

- $\mathrm{Gr}(k, N)$ – Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^N$ (compact manifold of dimension $k(N-k)$).
- $\gamma_{k,N}$ – the unique $O(N)$-invariant Borel probability measure on $\mathrm{Gr}(k, N)$.

  Equivalently: $\gamma_{k,N}$ is the distribution of $\mathrm{span}(l_1, \ldots, l_k)$, where $l_1, \ldots, l_k$ are independent random vectors, uniformly distributed in the unit ball $B_N(0,1)$ (or standard Gaussian).

  E.g. $\mathrm{Gr}(1,2) = \{\mathbb{R}(\cos\theta, \sin\theta) : \theta \in [0,\pi)\}$, $\gamma_{1,2} = \frac{1}{\pi}\mathrm{Leb}|_{[0,\pi)}$.
- $P_V \colon \mathbb{R}^N \to V \simeq \mathbb{R}^k$ – orthogonal projection onto $V \in \mathrm{Gr}(k, N)$.

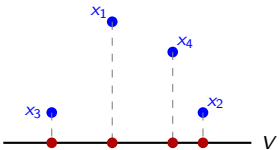
Given $X \subset \mathbb{R}^N$, when is P_V injective on X for $\gamma_{k,N}$-almost every $V \in \mathrm{Gr}(k, N)$?

Warm-up: countable sets

Example

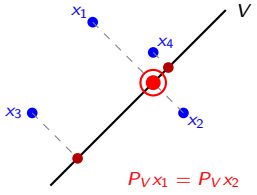
If $X \subset \mathbb{R}^N$ is countable, then P_V is injective on X for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$ (for every $k \geq 1$).

typical V



all projections distinct

bad $V = (x_1 - x_2)^\perp$



$P_V x_1 = P_V x_2$

Each pair $x_i \neq x_j$ has exactly one bad line $V = (x_i - x_j)^\perp$.

In general: P_V is injective on $X \iff V^\perp \cap (X - X) = \{0\}$.

Mañé's embedding theorem

Theorem (Mañé 1981)

Let $X \subset \mathbb{R}^N$ satisfy $\mathcal{H}^k(X - X) = 0$; in particular it suffices that

$$\dim_{\mathrm{H}}(X - X) < k \qquad \text{or} \qquad 2\overline{\dim}_{\mathrm{B}} X < k.$$

Then P_V is injective on X for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$.

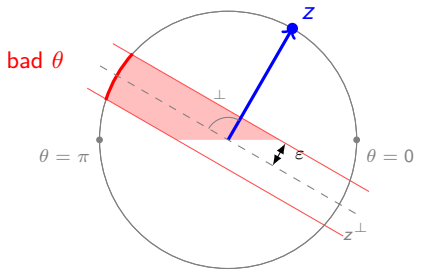
The proof is based on a quantitative estimate: the **transversality inequality**.

The key estimate - transversality

Lemma (Transversality inequality)

For every $z \in \mathbb{R}^N \setminus \{0\}$ and $\varepsilon > 0$

$$\gamma_{k,N}\big(\{V \in \mathrm{Gr}(k,N) : |P_V z| \le \varepsilon\}\big) \le C_{N,k}\big(\varepsilon/|z|\big)^k.$$



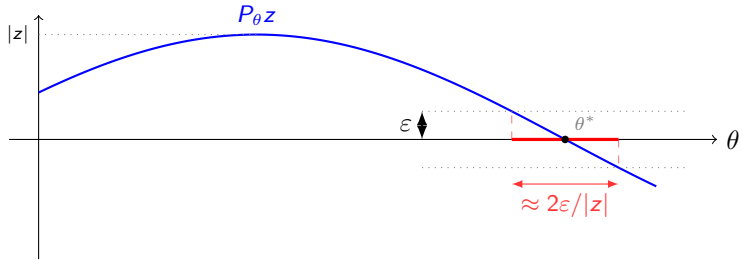
$k = 1$, $N = 2$, $|z| = 1$: $V_\theta = \mathbb{R}e_\theta$, $e_\theta = (\cos\theta, \sin\theta)$.

$|P_{V_\theta} z| = |\langle e_\theta, z\rangle| \le \varepsilon \iff e_\theta$ lies in the strip of half-width ε around $z^\perp$.

Bad θ: an interval of length $2\arcsin\varepsilon \le \pi\varepsilon$.

Why *transversality*?

For $N = 2,\ k = 1$, graph of $\theta \mapsto P_\theta z$ crosses zero *transversally*.



$P_\theta z = \langle z, e_\theta \rangle$ vanishes at θ^* with $e_{\theta^*} \perp z$, and $\frac{d}{d\theta} P_\theta z = \pm|z| \neq 0$ there.

For k-dimensional subspaces, as V varies, $P_V z$ moves in k independent directions with speed $\approx |z|$, so

$$\gamma_{k,N}\big(\{V : |P_V z| \leq \varepsilon\}\big) \approx (\varepsilon/|z|)^k.$$

Transversality will reappear for linear maps, delay-coordinate maps and families of IFS.

How sharp is Mañé's theorem?

Example (The factor 2 is necessary)

For the n-skeleton X of the $(2n+2)$-simplex: $\dim_{\mathrm{H}} X = \overline{\dim}_{\mathrm{B}} X = n$, but no continuous injective map $X \to \mathbb{R}^{2n}$ exists. E.g. no linear map $\mathbb{R}^3 \to \mathbb{R}^2$ is injective on $K_5 \subset \mathbb{R}^3$.

- Optimality for a given X: for $\overline{\dim}_{\mathrm{B}} X \le k < 2\overline{\dim}_{\mathrm{B}} X$ the answer depends on the geometry of X (segment vs. circle in $\mathbb{R}^2$, $k = 1$).

Example ($k > 2\dim_{\mathrm{H}} X$ is not sufficient)

There exist compact $X, Y \subset [0,1]$ with $\dim_{\mathrm{H}} X = \dim_{\mathrm{H}} Y = 0$ and $X + Y \supset [0,1]$. Hence $Z = X \cup (-Y)$ has

$$\dim_{\mathrm{H}} Z = 0, \qquad \dim_{\mathrm{H}}(Z - Z) = 1.$$

This can be used to construct, for every $m \ge 2$, a compact set $K \subset \mathbb{R}^m$ with $\dim_{\mathrm{H}} K = 0$ such that no linear map $L\colon \mathbb{R}^m \to \mathbb{R}^k$ with $k < m$ is injective on K. In particular, $2\dim_{\mathrm{H}} X < k$ is *not* sufficient in Mañé's theorem.

- Mañé (1981) stated the result under $k > 2\dim_{\mathrm{H}} X + 1$ – this is not sufficient.

Mañé's theorem for measures

Definition

$\phi\colon X \to \mathbb{R}^k$ is *μ-almost surely injective* if there is a Borel set $X_\phi \subset X$ with $\mu(X \setminus X_\phi) = 0$ such that $\phi|_{X_\phi}$ is injective.

Theorem (Probabilistic embedding theorem)

Let $X \subset \mathbb{R}^N$ be a Borel set with $\mathcal{H}^k(X) = 0$ and let μ be a σ-finite Borel measure on X. Then P_V is μ-a.s. injective for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$.

Corollary

Let μ be a finite Borel measure on $\mathbb{R}^N$ with $\mu \perp \mathcal{H}^k$ (e.g. $\dim_{\mathrm{H}} \mu < k$). Then P_V is μ-a.s. injective for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$.

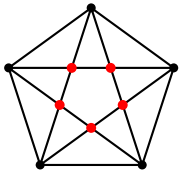
Note: we have actually proved that uf $\mathcal{H}^k(X) = 0$, then for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$, μ-a.e. $x \in X$ has the property

$$P_V x \neq P_V y \text{ **for all** } y \in X \setminus \{x\}$$

Why the factor 2 disappears

- **Sets:** one has to cover the set of pairs, $X - X$ ($k > 2\overline{\dim}_{\mathrm{B}}(X) \geq \overline{\dim}_{\mathrm{B}}(X - X)$).
- **Measures:** by Fubini, fix one point x and cover only $x - X$ ($k > \dim_{\mathrm{H}}(x - X) = \dim_{\mathrm{H}} X$).

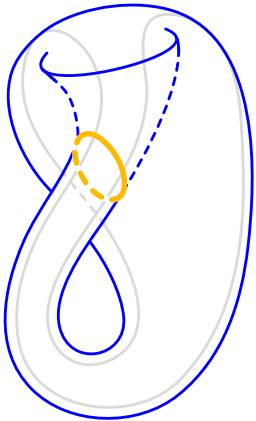
sets:	$k > 2\overline{\dim}_{\mathrm{B}} X$
measures:	$k > \dim_{\mathrm{H}} \mu$



$K_5 \to \mathbb{R}^2$: not injective, but the self-intersections form a set of length measure zero.

Almost sure embedding of the Klein bottle

- The Klein bottle K is a compact surface ($\dim K = 2$) which embeds into $\mathbb{R}^4$, but not into $\mathbb{R}^3$.
- For $K \subset \mathbb{R}^4$: a typical projection to $\mathbb{R}^3$ is injective outside a set of zero area (probabilistic embedding theorem with $\mu = \mathcal{H}^2|_K$, as $\dim_{\mathrm{H}} K = 2 < 3$).
- Picture: projection of $K \subset \mathbb{R}^4$ to $\mathbb{R}^3$, injective outside the self-intersection circle.

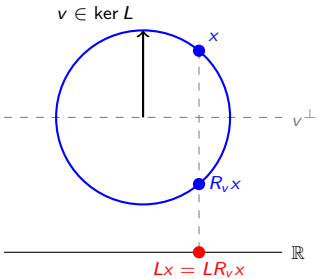


Sharpness: $\dim_{\mathrm{H}} \mu = k$ is not enough

Example

Let σ be the normalized length measure on the circle $S^1 \subset \mathbb{R}^2$ ($\dim_{\mathrm{H}} \sigma = 1$).

1. No linear map $L\colon \mathbb{R}^2 \to \mathbb{R}$ is σ-a.s. injective.
2. No Lipschitz map $\phi\colon S^1 \to \mathbb{R}$ is σ-a.s. injective.
3. There exists a continuous map $\psi\colon S^1 \to \mathbb{R}$ which is σ-a.s. injective.



Question: is $\dim_{\mathrm{H}} \mu$ the exact threshold for typical a.s. injectivity? $\rightsquigarrow$ conditional measures and projection theorems.

Conditional measures

Definition

Let $X \subset \mathbb{R}^N$ be compact, μ a Borel probability measure on X, $\phi\colon X \to \mathbb{R}^k$ continuous. $\{\mu_y\}_{y\in\mathbb{R}^k}$ is a *system of conditional measures* of μ w.r.t. ϕ if

1. μ_y is a (possibly zero) Borel measure concentrated on $\phi^{-1}(y)$,
2. μ_y is a probability measure for $\phi\mu$-a.e. y,
3. for every Borel $A \subset X$: $y \mapsto \mu_y(A)$ is measurable and $\mu(A) = \displaystyle\int_{\mathbb{R}^k} \mu_y(A)\, d\phi\mu(y)$.

- Exists and is unique up to a $\phi\mu$-null set of y's (Rokhlin's disintegration theorem).
- Simmons: for $\phi\mu$-a.e. y,

$$\mu_y = \lim_{\varepsilon\to 0} \frac{\mu|_{\phi^{-1}(B(y,\varepsilon))}}{\mu(\phi^{-1}(B(y,\varepsilon)))} \qquad \text{(weak-}*\text{)}.$$

A.s. injectivity via conditional measures

Lemma

Let $X \subset \mathbb{R}^N$ be compact, μ a Borel probability measure on X, $\phi\colon X \to \mathbb{R}^k$ continuous, and $\{\mu_y\}$ the conditional measures of μ w.r.t. ϕ. The following are equivalent:

(a) *ϕ is μ-a.s. injective,*

(b) *$\mu_{\phi(x)} = \delta_x$ for μ-a.e. $x \in X$,*

(c) *μ_y is a Dirac measure for $\phi\mu$-a.e. $y \in \mathbb{R}^k$.*

Projection and slicing theorems

Theorem

Let $X \subset \mathbb{R}^N$ be Borel and μ a finite Borel measure on $\mathbb{R}^N$. For $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$:

1. $\dim_{\mathrm{H}} P_V X = \min\{k, \dim_{\mathrm{H}} X\}$, *and* $\mathcal{H}^k(P_V X) > 0$ *if* $\dim_{\mathrm{H}} X > k$;
2. $\dim_{\mathrm{H}} P_V \mu = \min\{k, \dim_{\mathrm{H}} \mu\}$, $\underline{\dim}_{\mathrm{H}} P_V \mu = \min\{k, \underline{\dim}_{\mathrm{H}} \mu\}$, *and* $P_V \mu \ll \mathcal{H}^k$ *if* $\underline{\dim}_{\mathrm{H}} \mu > k$;
3. *if μ is a compactly supported probability measure with $\underline{\dim}_{\mathrm{H}} \mu > k$, then the conditional measures μ_a^V of μ w.r.t. P_V satisfy*

$$\underline{\dim}_{\mathrm{H}} \mu_a^V \geq \underline{\dim}_{\mathrm{H}} \mu - k \qquad \text{for } P_V \mu\text{-a.e. } a \in V.$$

$\dim_{\mathrm{H}} \mu$ is the threshold

Corollary

Let μ be a finite Borel measure on $\mathbb{R}^N$ with $\dim_{\mathrm{H}} \mu > k$. Then for $\gamma_{k,N}$-a.e. V, P_V is not injective on any Borel set of full μ-measure.

Hence, for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$:

$$\dim_{\mathrm{H}} \mu < k \implies P_V \text{ is } \mu\text{-a.s. injective}, \qquad \dim_{\mathrm{H}} \mu > k \implies P_V \text{ is not } \mu\text{-a.s. injective.}$$

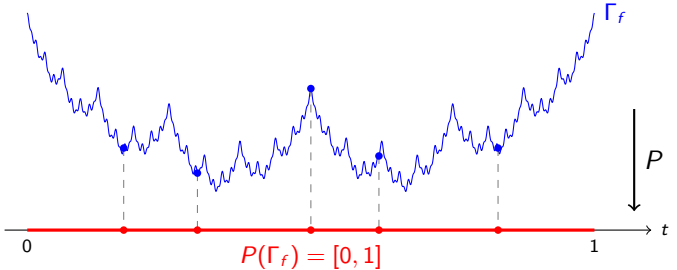
Remark (Border case $\dim_{\mathrm{H}} \mu = k$)

Both behaviours occur: $\mathcal{H}^1$ on a segment in $\mathbb{R}^2$ (a.s. injective) vs. $\mathcal{H}^1$ on a circle (not).
Related to an open problem of Mattila on purely unrectifiable 1-sets.

Injectivity does not prevent dimension drop

Example

Weierstrass function $f(t) = \sum_{n\geq 0} \lambda^n \cos(2\pi b^n t)$, $1/b < \lambda < 1$: $\dim_{\mathrm{H}} \Gamma_f = 2 + \frac{\log \lambda}{\log b} > 1$. The projection $P(t, y) = t$ is injective on Γ_f, but $\dim_{\mathrm{H}} P(\Gamma_f) = 1 < \dim_{\mathrm{H}} \Gamma_f$.



Taking instead a graph Γ_f over a Cantor set of small dimension, one can have $\overline{\dim}_{\mathrm{B}} \Gamma_f < \frac{1}{2}$, so that Mañé's theorem applies; still P is injective on Γ_f and $\dim_{\mathrm{H}} P(\Gamma_f) < \dim_{\mathrm{H}} \Gamma_f$.