# Embedding theory in geometric measure theory

## Part 2: Regularity of the embedding

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# Recall: Mañé's theorems

**Orthogonal projections** $P_V : \mathbb{R}^N \to V \simeq \mathbb{R}^k$, $V \in \mathrm{Gr}(k, N)$, with the measure $\gamma_{k,N}$.

**Theorem** (Mañé 1981)**.** Let $X \subset \mathbb{R}^N$ satisfy $\mathcal{H}^k(X - X) = 0$. Then $P_V$ is injective on $X$ for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$.
In particular it suffices that $\dim_{\mathrm{H}}(X - X) < k$ or $2\overline{\dim}_{\mathrm{B}} X < k$.

**Theorem** (Probabilistic embedding theorem)**.** Let $X \subset \mathbb{R}^N$ be a Borel set with $\mathcal{H}^k(X) = 0$ and let $\mu$ be a $\sigma$-finite Borel measure on $X$. Then $P_V$ is $\mu$-a.s. injective for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$.
In particular, it suffices that $\dim_{\mathrm{H}} \mu < k$.

If $\dim_{\mathrm{H}} \mu > k$, then for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$, the map $P_V$ is not $\mu$-a.s. injective.

# The same holds for a.e. linear map

$\mathrm{Lin}(\mathbb{R}^N, \mathbb{R}^k) := \{\text{linear maps} \mathbb{R}^N \to \mathbb{R}^k\} \simeq \mathbb{R}^{kN}$ with the Lebesgue measure.

Mañé's theorems (and all the results below) hold for Lebesgue-a.e. $L \in \mathrm{Lin}(\mathbb{R}^N, \mathbb{R}^k)$.

- A.e. $L \in \mathrm{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ has **full rank** and can be written uniquely as $L = M \circ P_V$, where $V = (\ker L)^\perp \in \mathrm{Gr}(k, N)$ and $M \colon V \to \mathbb{R}^k$ is a linear isomorphism.

# Why bother? Compressed sensing and beyond

- **Compressed sensing:** a vector $x \in \mathbb{R}^N$ with at most $s$ non-zero coordinates can be recovered from $k \approx s \log(N/s)$ random linear measurements $Lx \in \mathbb{R}^k$; effective decoding by $\ell^1$-minimization (Candès–Romberg–Tao, Donoho 2006).
- **Embedding point of view:** $X =$ finite union of $s$-dimensional linear subspaces, $\dim X = s \Rightarrow$ a.e. $L\colon \mathbb{R}^N \to \mathbb{R}^k$ with $k > 2s$ is injective on $X$ (Mañé).
- **Goal**: relax sparsity to constraints on the geometric complexity of the signal
- Wu–Verdú 2010 initiated study of **analog compression** of stochastic processes with complexity measured as mean (average) dimension of the process:
- Signal $(X_i)_{i=1}^{\infty}$ (stochastic process), compressed by linear measurements of $(X_1, \dots, X_n)$ into $\mathbb{R}^{k_n}$. Characterize minimal measurement rate $\lim_{n\to\infty} \frac{k_n}{n}$ for (almost) lossless recovery?
- For rapidly mixing processes with zero mean-square error compression: **mean information dimension** (Jalali–Poor 2017, Gutman–Ś 2026 (optimality, GMT methods)).

# Regularity of the inverse

- **Compression** $=$ applying $\phi$ (e.g. a linear map $L$), **decompression** $=$ the inverse $\phi^{-1}$ on the set of injectivity.
- **With noise:** if $\phi^{-1}$ is $\alpha$-Hölder with constant $C$ and $\phi(x)$ is known up to $\varepsilon$, then $x$ is recovered up to $C\varepsilon^{\alpha}$.

**Does the inverse have a quantitative modulus of continuity?**

The answer depends on the notion of dimension.

# Hölder inverses

## Definition

Let $(X, \rho)$ be a metric space, $\phi\colon X \to \mathbb{R}^k$ a Lipschitz map and $0 < \alpha \leq 1$.

1. $\phi$ has an **$\alpha$-Hölder inverse** on $X$ if there is $C > 0$ with $\rho(x, y) \leq C|\phi(x) - \phi(y)|^\alpha$ for all $x, y \in X$. If $\phi$ has a Lipschitz inverse, then $\phi$ is a **bi-Lipschitz embedding**.
2. $\phi$ has a **pointwise $\alpha$-Hölder inverse at $x \in X$** if there is $C_x > 0$ with $\rho(x, y) \leq C_x|\phi(x) - \phi(y)|^\alpha$ for all $y \in X$.
3. Let $\mu$ be a finite Borel measure on $X$. The map $\phi$ is **$\mu$-nearly bi-Lipschitz** if for $\mu$-a.e. $x$ it has a pointwise $\alpha$-Hölder inverse at $x$ for every $\alpha \in (0, 1)$.

A pointwise Hölder inverse at every point of $X' \subset X$ implies injectivity on $X'$.

# The regularity theorem: sets

Let $X \subset \mathbb{R}^N$ be compact.

**Theorem (A)** (Uniform regularity)**.** For $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$:

(i) if $\dim_{\mathrm{H}}(X - X) < k$, then $P_V$ is injective on $X$, so it has a continuous inverse;

(ii) if $2\overline{\dim}_{\mathrm{B}} X < k$, then $P_V$ has an $\alpha$-Hölder inverse on $X$ for every $\alpha \in \left(0, 1 - \frac{2\overline{\dim}_{\mathrm{B}} X}{k}\right)$;

(iii) if $\dim_{\mathrm{A}}(X - X) < k$, then $P_V$ has an $\alpha$-Hölder inverse on $X$ for every $\alpha \in (0, 1)$.

- (ii): Ben-Artzi–Eden–Foias–Nicolaenko 1993, Hunt–Kaloshin 1999; (iii): Olson 2002.
- In (iii), $\dim_{\mathrm{A}}(X - X) < k$ cannot be replaced by $2 \dim_{\mathrm{A}} X < k$: $\dim_{\mathrm{A}}$ may increase under Lipschitz maps.

# The regularity theorem: measures

**Theorem (B)** (Pointwise regularity)**.** Let $\mu$ be a finite Borel measure on $\mathbb{R}^N$ with compact support $X$. Then for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$:

(i) if $\dim_{\mathrm{H}} \mu < k$, then $P_V$ is $\mu$-a.s. injective; if moreover $\dim_{\mathrm{H}} X < k$, then for $\mu$-a.e. $x$ the inverse is continuous at $P_V(x)$;

(ii) if $\overline{\dim}_{\mathrm{B}} X < k$, then for $\mu$-a.e. $x$ the map $P_V$ has a pointwise $\alpha$-Hölder inverse at $x$ for every $\alpha \in \left(0, 1 - \frac{\overline{\dim}_{\mathrm{B}} X}{k}\right)$;

(iii) if $\dim_{\mathrm{A}} X < k$, then $P_V$ is $\mu$-nearly bi-Lipschitz.

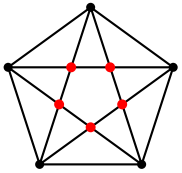
# No uniform Hölder inverse for measures

## Remark

In (B) one cannot ask for a **uniformly** Hölder inverse on a set of full measure.

E.g. $X = K_5 \subset \mathbb{R}^3$, $\mu =$ length measure, $k = 2$:

- $\dim_{\mathrm{A}} X = 1 < 2$, so $P_V$ is $\mu$-nearly bi-Lipschitz for a.e. $V \in \mathrm{Gr}(2, 3)$;
- but no continuous $\phi\colon X \to \mathbb{R}^2$ has a uniformly Hölder inverse on a set $X_\phi$ of full measure: it would extend to $\overline{X_\phi} = X$, giving an embedding $K_5 \hookrightarrow \mathbb{R}^2$.



# Sharpness

## Remark

The assumptions in (B) are **sharp**:

- in (i), $\dim_{\mathrm{H}} X$ cannot be replaced by $\dim_{\mathrm{H}} \mu$;
- in (ii), $\overline{\dim}_{\mathrm{B}} X$ cannot be replaced by $\dim_{\mathrm{H}} X$, and the range of exponents is optimal;
- in (iii), $\dim_{\mathrm{A}} X$ cannot be replaced by $\overline{\dim}_{\mathrm{B}} X$.
- In (iii), $P_V$ need not be pointwise bi-Lipschitz on any set of full $\mu$-measure, even if $\dim_{\mathrm{A}} X = 0$

In (iii) one gets in fact a Lipschitz inverse up to a logarithmic factor: if $\gamma > \frac{1}{k - \dim_{\mathrm{A}} X}$, then for $\mu$-a.e. $x$ there is $C_x > 0$ with

$$|P_V x - P_V y| \geq C_x \frac{|x-y|}{\log\left(2 \operatorname{diam} X / |x-y|\right)^{\gamma}} \qquad \text{for all } y \in X.$$

# Covering estimate

**Transversality inequality:** for $z \in \mathbb{R}^N \setminus \{0\}$ and $\varepsilon > 0$,

$$\gamma_{k,N}\big(\{V \in \mathrm{Gr}(k,N) : |P_V z| \le \varepsilon\}\big) \le C_{N,k}\Big(\frac{\varepsilon}{|z|}\Big)^k.$$

## Lemma

*Let $X \subset \mathbb{R}^N$ be compact and $x \in X$. For $0 < \varepsilon < \delta/2$ let*

$$B_x(\varepsilon,\delta) = \{V \in \mathrm{Gr}(k,N) : |P_V x - P_V y| \le \varepsilon \text{ for some } y \in X \text{ with } |x-y| \ge \delta\}.$$

*For every $\theta > 0$ there is $D = D(X,\theta) > 0$, independent of $x, \varepsilon, \delta$, such that*

(a) $\gamma_{k,N}(B_x(\varepsilon,\delta)) \le D\,\varepsilon^{k-\overline{\dim}_{\mathrm{B}} X-\theta}\,\delta^{-k}$;

(b) *if* $\dim_{\mathrm{A}} X + \theta < k$, *then* $\gamma_{k,N}(B_x(\varepsilon,\delta)) \le D\,(\varepsilon/\delta)^{k-\dim_{\mathrm{A}} X-\theta}$.

**We have**

Lemma

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(a) $\gamma_{k,N}(B_x(\varepsilon, \delta)) \leq D\, \varepsilon^{k - \overline{\dim}_{\mathrm{B}} X - \theta}\, \delta^{-k}$;

(b) *if* $\dim_{\mathrm{A}} X + \theta < k$, *then* $\gamma_{k,N}(B_x(\varepsilon, \delta)) \leq D\, (\varepsilon/\delta)^{k - \dim_{\mathrm{A}} X - \theta}$.

**We prove**

**Theorem (B)** (Pointwise regularity)**.** Let $\mu$ be a finite Borel measure on $\mathbb{R}^N$ with compact support $X$. Then for $\gamma_{k,N}$-a.e. $V \in \mathrm{Gr}(k, N)$:

(ii) if $\overline{\dim}_{\mathrm{B}} X < k$, then for $\mu$-a.e. $x$ the map $P_V$ has a pointwise $\alpha$-Hölder inverse at $x$ for every $\alpha \in \left(0, 1 - \frac{\overline{\dim}_{\mathrm{B}} X}{k}\right)$;

(iii) if $\dim_{\mathrm{A}} X < k$, then $P_V$ is $\mu$-nearly bi-Lipschitz.

# Prevalence

There is no Lebesgue measure on infinite-dimensional spaces. Replacement for "almost every":

## Definition (Christensen 1972; Hunt–Sauer–Yorke 1992)

Let $V$ be a complete metrizable topological vector space. A Borel set $S \subset V$ is **prevalent** if there is a Borel measure $\nu$ on $V$ with $0 < \nu(K) < \infty$ for some compact $K$, such that for every $v \in V$, one has $v + e \in S$ for $\nu$-a.e. $e$.

- In $\mathbb{R}^n$: prevalent $\iff$ full Lebesgue measure. Prevalent sets are dense and closed under countable intersections.

**Prevalent versions.** Fix a Lipschitz map $\phi\colon X \to \mathbb{R}^k$. One can prove versions of Mañé's theorems and regularity theorems for $\phi + L$ for a.e. $L \in \mathrm{Lin}(\mathbb{R}^N, \mathbb{R}^k)$. Therefore they hold for prevalent Lipschitz maps $X \to \mathbb{R}^k$.

# Summary of Parts 1 and 2

| **object** | **assumption** | **conclusion for $\gamma_{k,N}$-a.e. $V$** |
|---|---|---|
| set $X$ | $\dim_{\mathrm{H}}(X-X) < k$ | continuous inverse |
| set $X$ | $2\overline{\dim}_{\mathrm{B}} X < k$ | $\alpha$-Hölder inverse, $\alpha < 1 - \frac{2\overline{\dim}_{\mathrm{B}} X}{k}$ |
| set $X$ | $\dim_{\mathrm{A}}(X-X) < k$ | $\alpha$-Hölder inverse, $\alpha < 1$ |
| measure $\mu$ | $\dim_{\mathrm{H}} \mu > k$ | not a.s. injective |
| measure $\mu$ | $\dim_{\mathrm{H}} \mu < k$ | a.s. injective |
| measure $\mu$ | $\dim_{\mathrm{H}} \operatorname{supp} \mu < k$ | a.s. injective, continuous inverse |
| measure $\mu$ | $\overline{\dim}_{\mathrm{B}} \operatorname{supp} \mu < k$ | pointwise $\alpha$-Hölder, $\alpha < 1 - \frac{\overline{\dim}_{\mathrm{B}} \operatorname{supp} \mu}{k}$ |
| measure $\mu$ | $\dim_{\mathrm{A}} \operatorname{supp} \mu < k$ | nearly bi-Lipschitz |