# Embedding theory in geometric measure theory

## Part 4: Applications to fractal geometry

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# Iterated function systems

## Definition

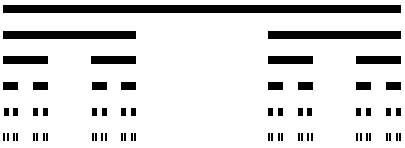
An **iterated function system** (IFS) on a closed set $V \subset \mathbb{R}^d$ is a finite family $\mathcal{F} = \{f_i\}_{i \in \mathcal{A}}$ of contractions $f_i \colon V \to V$.

**Theorem** (Hutchinson 1981)**.** There is a unique non-empty compact set $\Lambda = \Lambda_{\mathcal{F}} \subset V$ (the attractor) with

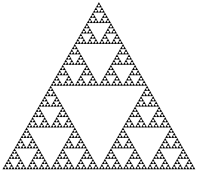
$$\Lambda = \bigcup_{i \in \mathcal{A}} f_i(\Lambda).$$

- We will consider **Self-similar** IFS: $f_i(x) = r_i O_i x + t_i$ with $r_i \in (0,1)$, $O_i \in O(d)$, $t_i \in \mathbb{R}^d$.
- Almost everything below extends to **self-conformal** IFS ($C^{1+\theta}$ conformal maps).

# Examples



Middle-third Cantor set:
$\{x \mapsto x/3,\ x \mapsto x/3 + 2/3\}$ on $\mathbb{R}$.



Sierpiński gasket:
$\{x \mapsto x/2 + t_i\}_{i=1,2,3}$ on $\mathbb{R}^2$.

**Separation:** $\mathcal{F}$ satisfies the **strong separation condition** (SSC) if the pieces $f_i(\Lambda)$ are pairwise disjoint. Then $\dim_{\mathrm{H}} \Lambda = s(\mathcal{F})$, where $\sum_i r_i^{s(\mathcal{F})} = 1$.

**Overlapping cases:** difficult

# Symbolic coding

$\Sigma = \mathcal{A}^{\mathbb{N}}$. For word $\omega = \omega_1\omega_2\ldots$ let $\omega|_n = \omega_1\ldots\omega_n$ and $f_{\omega|_n} = f_{\omega_1} \circ \cdots \circ f_{\omega_n}$

## Lemma (Natural projection)

*For every $\omega \in \Sigma$ and $x_0 \in V$ the limit*

$$\pi(\omega) = \pi_{\mathcal{F}}(\omega) = \lim_{n\to\infty} f_{\omega|_n}(x_0)$$

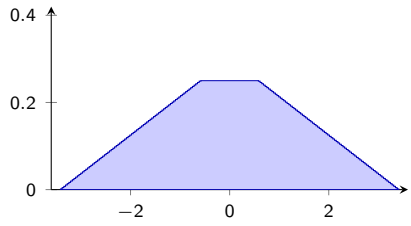
*exists and does not depend on $x_0$. The map $\pi\colon \Sigma \to \mathbb{R}^d$ is continuous and $\pi(\Sigma) = \Lambda$.*

- $\pi$ is injective iff SSC holds
- **Measures:** for a probability vector $p$, the image $\nu_p = \pi p^{\mathbb{N}}$ of the Bernoulli measure is the **self-similar measure**, $\nu_p = \sum_i p_i f_i \nu_p$.

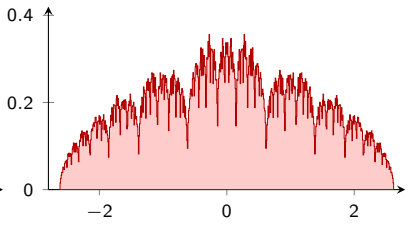
# Bernoulli convolutions

$\mathcal{F}_\lambda = \{x \mapsto \lambda x - 1,\ x \mapsto \lambda x + 1\}$, $p = (\frac{1}{2}, \frac{1}{2})$: $\nu_\lambda$ = distribution of $\sum_{n \geq 0} \pm \lambda^n$.

$\lambda = 1/\sqrt{2}$




$\lambda = (\sqrt{5} - 1)/2$




- For $\lambda \in [\frac{1}{2}, 1)$: attractor is an interval, but $\nu_\lambda$ can be complicated

# Heuristics: dimension = entropy / Lyapunov exponent

Let $f_i(x) = r_i O_i x + t_i$ satisfy the SSC, $\mu = p^{\mathbb{N}}$ and $\nu = \pi\mu$

- **Cylinders** $f_{\omega|_n}(\Lambda) \ni \pi(\omega)$ have explicit measure and size:

$$\nu\big(f_{\omega|_n}(\Lambda)\big) = \mu[\omega|_n] = p_{\omega_1} \cdots p_{\omega_n}, \qquad \operatorname{diam} f_{\omega|_n}(\Lambda) = r_{\omega_1} \cdots r_{\omega_n} \operatorname{diam} \Lambda.$$

- By the SSC, at scale $r \approx \operatorname{diam} f_{\omega|_n}(\Lambda)$ the interection $\Lambda \cap B(\pi(\omega), r)$ is essentially the cylinder $f_{\omega|_n}(\Lambda)$.

Hence, by the **strong law of large numbers**, for $\mu$-a.e. $\omega$

$$\frac{\log \nu\big(B(\pi(\omega), r)\big)}{\log r} \approx \frac{\log \mu[\omega|_n]}{\log(r_{\omega_1} \cdots r_{\omega_n})} = \frac{\frac{1}{n}\sum_{j=1}^n \log p_{\omega_j}}{\frac{1}{n}\sum_{j=1}^n \log r_{\omega_j}} \xrightarrow{n\to\infty} \frac{-\sum_i p_i \log p_i}{-\sum_i p_i \log r_i} = \frac{h(\mu)}{\chi(\mu)}.$$

**Entropy** $h(\mu)$: exponential rate of decay of $\mu[\omega|_n]$
**Lyapunov exponent** $\chi(\mu)$: exponential rate of decay of $\operatorname{diam} f_{\omega|_n}(\Lambda)$.

# The symbolic metric

For $\mathcal{F} = \{r_i O_i x + t_i\}$ let $r_u = r_{u_1} \cdots r_{u_n}$ and for the longest common prefix $\omega \wedge \tau$ set

$$\rho_{\mathcal{F}}(\omega, \tau) = r_{\omega \wedge \tau} \quad (\omega \neq \tau)$$

## Proposition

1. $\pi \colon (\Sigma, \rho_{\mathcal{F}}) \to \mathbb{R}^d$ *is Lipschitz; it is injective iff SSC holds, and then bi-Lipschitz.*
2. *Every ergodic* $\mu$ *on* $\Sigma$ *is exact dimensional with* $\dim(\mu, \rho_{\mathcal{F}}) = \dfrac{h(\mu)}{\chi_{\mathcal{F}}(\mu)}$.
3. $\dim_{\mathrm{H}}(\Sigma, \rho_{\mathcal{F}}) = \dim_{\mathrm{A}}(\Sigma, \rho_{\mathcal{F}}) = s(\mathcal{F})$.

Hence $\dim_{\mathrm{H}} \pi\mu \le \min\{d, h(\mu)/\chi_{\mathcal{F}}(\mu)\}$ and $\dim_{\mathrm{H}} \Lambda \le \min\{d, s(\mathcal{F})\}$, with equality under SSC.

# Natural projections as embeddings

$\pi\colon (\Sigma, \rho_{\mathcal{F}}) \to \mathbb{R}^d$ is a Lipschitz map from a space of dimension $s(\mathcal{F})$;
$\mu \in \mathcal{E}(\Sigma)$ has dimension $h(\mu)/\chi_{\mathcal{F}}(\mu)$.

**Is $\pi$ ($\mu$-a.s.) injective? How regular is the inverse?**

**Theorem** (Feng–Hu 2009)**.** Let $\mathcal{F}$ be self-similar and $\mu \in \mathcal{E}(\Sigma)$. Then $\pi\mu$ is exact dimensional, and

$$\pi \text{ is } \mu\text{-a.s. injective} \iff \dim \pi\mu = \frac{h(\mu)}{\chi_{\mathcal{F}}(\mu)}.$$

For general maps this fails: injective Lipschitz maps can lower the dimension. The self-similar structure is crucial.

# Parametrized families and transversality

Family $\{\mathcal{F}^\lambda\}_{\lambda\in U}$ of self-similar IFS, $\pi_\lambda = \pi_{\mathcal{F}^\lambda}$, finite measure $\eta$ on $U$.

## Definition

The family satisfies the **transversality condition** if there is $C > 0$ with

$$\eta\big(\{\lambda \in U : |\pi_\lambda(\omega) - \pi_\lambda(\tau)| \le r\}\big) \le Cr^d \quad \text{for all } r > 0,\ \omega_1 \ne \tau_1.$$

- Analogue of the transversality inequalities of Parts 1–3 – but it **does not come for free**.
- **Examples:**
  - orthogonal projections $P_V \circ \pi_{\mathcal{F}}$ of a self-similar set with SSC;
  - translation families $\{r_i O_i x + t_i\}$, parameter $t = (t_i)$, if $\max_{i\ne j}(r_i + r_j) < 1$; on $\mathbb{R}$ for $x \mapsto r_i x + t_i$ whenever $s(\mathcal{F}) < 1$;
  - Bernoulli convolutions for $\lambda \in (0, 0.649]$; it fails on $(2^{-1/2}, 1)$.

# Universal projection theorem

**Theorem** (Bárány–Simon–Ś 2025)**.** Let $\{\mathcal{F}^\lambda\}_{\lambda\in U}$ be a continuous family of self-similar IFS on $\mathbb{R}^d$ satisfying the transversality condition. Then for $\eta$-a.e. $\lambda$, the following holds for all $\mu \in \mathcal{E}(\Sigma)$:

(i) $\dim_{\mathrm{H}} \pi_\lambda \mu = \min\{d, h(\mu)/\chi_\lambda(\mu)\}$;

(ii) if $s(\mathcal{F}^\lambda) < d$, then $\pi_\lambda$ is $\mu$-nearly bi-Lipschitz.

- For a fixed $\mu$, (i) is classical (Simon–Solomyak–Urbański 2001) – analogue of the probabilistic embedding theorem.
- (ii) is the analogue of the regularity theorem (B)(iii), since $s(\mathcal{F}^\lambda) = \dim_{\mathrm{A}}(\Sigma, \rho_\lambda)$.
- **Novelty: universality** – the exceptional set of parameters does not depend on $\mu$; $\mathcal{E}(\Sigma)$ is uncountable.

# Application: multifractal analysis

A self-similar measure is exact dimensional, but other scaling exponents occur on null sets, which can be large.

## Definition

For a finite Borel measure $\nu$ and $\alpha \geq 0$:

$$E_\nu(\alpha) = \{x \in \operatorname{supp} \nu : \underline{d}(\nu, x) = \overline{d}(\nu, x) = \alpha\},$$

$$f_\nu(\alpha) = \dim_{\mathrm{H}} E_\nu(\alpha) \quad (\textbf{multifractal spectrum}),$$

# Self-similar measures on $\mathbb{R}$

$\nu = \pi_F p^{\mathbb{N}}$ for $\mathcal{F} = \{x \mapsto r_i x + t_i\}_{i \in \mathcal{A}}$. Define $T(q)$ and its Legendre transform by

$$\sum_{i \in \mathcal{A}} p_i^q |r_i|^{-T(q)} = 1, \qquad T^*(\alpha) = \inf_{q \in \mathbb{R}} (\alpha q - T(q))$$

- **SSC:** $f_\nu(\alpha) = T^*(\alpha)$ on $[\alpha_{\min}, \alpha_{\max}]$ (Cawley–Mauldin 1992, Arbeiter–Patzschke 1996), where
$$\alpha_{\min} = \min \frac{\log p_i}{\log |r_i|}, \ \alpha_{\max} = \max \frac{\log p_i}{\log |r_i|}$$
- **Overlaps:** Barral–Feng 2021: if $s_0 < 1$ and exponential separation holds (a.e. $t$), then $f_\nu = T^*$ on the **increasing part** $[\alpha_{\min}, \alpha_0]$, $\alpha_0 = T'(0)$.
- The **decreasing part** $(\alpha > \alpha_0)$ remained out of reach.

# Main result: full spectrum for typical translations

**Theorem** (Bárány–Simon–Ś 2025)**.** Fix $r_i \in (-1,1) \setminus \{0\}$, $i \in \mathcal{A}$, with $s_0 < 1$. Then for Lebesgue-a.e. $t \in \mathbb{R}^{\mathcal{A}}$,

$$\dim_{\mathrm{H}} E_{\nu_{t,p}}(\alpha) = T^*(\alpha)$$

for **every** probability vector $p \neq (|r_i|^{s_0})_i$ and **every** $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.

- **Upper bound** on $\dim_{\mathrm{H}} E_{\nu_{t,p}}(\alpha)$: standard.
- **Key fact for the lower bound:** if $\pi$ is $\mu$-nearly bi-Lipschitz, then at $\mu$-a.e. $\omega$

  $$d(\pi\nu, \pi(\omega)) = d(\nu, \omega) \qquad \text{for every probability measure } \nu \text{ on } \Sigma.$$

- **Idea of the lower bound:** there is a natural Bernoulli measure $\mu_\alpha$ on $\{\omega \in \Sigma : d(p^{\mathbb{N}}, \omega) = \alpha\}$ with $\dim \mu_\alpha = h(\mu_\alpha)/\chi(\mu_\alpha) = T^*(\alpha)$. By the universal projection theorem, $\dim \pi_t \mu_\alpha = T^*(\alpha)$ and by the nearly bi-Lipschitz property $\pi_t \mu_\alpha$ lives on $E_{\nu_{t,p}}(\alpha)$, so $\dim_H E_{\nu_{t,p}}(\alpha) \geq \dim \pi_t \mu_\alpha = T^*(\alpha)$.