Assignment

PhD Course in Probability and Statistics, Part I

Due date 28 March 2024, 23:59 CEST

Please submit your solutions either as a pdf by email to `sascha.troscheit@math.uu.se` by the due date/time or in person beforehand. Each problem counts 5 points and at least 45% on each of the two homework assignments is required to pass the assignments part of the course. Answers will be subject to a short (30min) oral exam at the end of the course. While you may discuss the questions with other students, all submitted solutions must be your own. If you use any sources other than the course book and lecture notes, this should be indicated clearly. You are most welcome to contact me (in person or by email: `sascha.troscheit@math.uu.se`) with any questions.

1. Prove the following.

   (a) Let (X_n) be a sequence of i.i.d. random variables with the standard normal distribution. Show that $Y_k = \min\{X_1, X_2, \dots, X_k\}$ satisfies $Y_k \to -\infty$ almost surely. (2)

   (b) Let $S_n = Z_1 + \cdots + Z_n$ be a simple biased random walk, i.e. with step size 1 and $\mathbb{P}(Z_i = 1) \neq \frac{1}{2}$. Show that

   $$\mathbb{P}(S_n = 0 \text{ for infinitely many } n) = 0$$

   (Hint: you can use Stirling's approximation if needed: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.) (2)

2. For each of the following statements, determine whether it is true or false. If true, give a proof. If false, provide a counterexample.

   (a) Let $\mathcal{F}_n$ be a filtration. Then $\bigcup_n \mathcal{F}_n$ is a σ algebra. (1)

   (b) Let M_n be a martingale that is pre-visible. Then M_n is, almost surely, constant. (1)

   (c) Let M_n be a martingale with $\mathbb{E}(M_n) = 0$ for all $n \in \mathbb{N}$. Then M_n converges to some M_∞ almost surely. (1)

   (d) Let X_n be a sequence of independent standard normal random variables. Let

   $$S_n = \sum_{i=1}^{n} X_n.$$

   Then, $\tau = \min\{n \mid S_n \geq 2S_{n-1}\}$ is a stopping time (wrt the natural filtration). (1)

3. Let $\Omega = \{1, 2, 3, 4, 5\}$ and $\Sigma = \mathcal{P}(\Omega)$ and let $\mathbb{P}$ be the probability measure given by

   $$p_i = \mathbb{P}(\{i\}) = \begin{cases} \frac{1}{4} & i \text{ is even,} \\ \frac{1}{6} & i \text{ is odd.} \end{cases}$$

and assume $p_i > 0$ for all $i \in \Omega$. Define

$$X_1(\omega) = 1, \quad X_2(\omega) = 1/\omega, \quad X_3(\omega) = \begin{cases} \omega^2 & \omega \text{ is odd} \\ 0 & \omega \text{ is even} \end{cases}$$

Determine

(a) $\mathbb{E}(X_1), \mathbb{E}(X_2), \mathbb{E}(X_3)$

(b) $\mathbb{E}(X_2 \mid X_3)$

(c) $\mathbb{E}(X_3 \mid X_2)$

(d) $\mathbb{E}(X_2 \mid X_1)$

(e) Describe the information that we obtain from the σ-algebras $\sigma(X_1), \sigma(X_2), \sigma(X_3)$.

4. Consider a sequence of independent identically distributed random variables $Y_1, Y_2, \ldots$, each following a uniform distribution on $[0, a]$. Set

$$X_n = Y_1 \cdot Y_2 \cdots Y_n.$$

(a) For which value of a is $X_1, X_2, \cdots$ a martingale?

(b) Show that there is a value a_0 such that the following holds: if $a > a_0$, then $X_n \to \infty$ almost surely, and if $a < a_0$, then $X_n \to 0$ almost surely. Determine a_0.

5. Let X_0 be a random variable that is uniform in $[-1, 1]$ and let

$$X_{n+1} = \begin{cases} \frac{5}{4}X_n + \frac{1}{4}X_n^2 - \frac{1}{4}X_n^3 - \frac{1}{4}X_n^4 & \text{with probability } \frac{1}{3}, \\ \frac{5}{6}X_n - \frac{1}{2}X_n^2 + \frac{1}{6}X_n^3 + \frac{1}{2}X_n^4 & \text{with probability } \frac{1}{3}, \\ \frac{11}{12}X_n + \frac{1}{4}X_n^2 + \frac{1}{12}X_n^3 - \frac{1}{4}X_n^4 & \text{with probability } \frac{1}{3}. \end{cases}$$

(a) Show that X_n converges almost surely.

(b) Determine the set of possible limits L.

(c) For each possible limit $a \in L$, determine $\mathbb{P}(X_\infty = a)$.

6. **Pólya's Urn.** At step 0, an urn contains $N(0) \geq 2$ many balls of which $N_w(0) \geq 1$ are white and $N_b(0) \geq 1$ are black. At time n we randomly draw a ball from the urn and replace it. Additionally, we include another ball of the same colour. Show that the proportion of white balls $N_w(n)/N(n)$ at time n is a martingale.

7. Let Y_k be a sequence of i.d.d. random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$ such that Y_k is not a constant almost surely. Let $Z_k = \sum_{i=1}^{k} Y_i$ be their random sum.

(a) Show that the sequence of σ algebras $\mathcal{A}_n = \sigma\langle Y_1, \ldots, Y_n\rangle$ is a filtration.

(b) Show that $\sigma\langle Z_1, \ldots, Z_k\rangle = \mathcal{A}_k$ but $\sigma\langle Z_k\rangle \neq \mathcal{A}_k$.

(c) Assume $\mathbb{E}Y = 0$. Is $\mathbb{E}(Z_{k+1}|\mathcal{A}_k)$ a martingale? Is $\mathbb{E}(Z_{k+1}|\sigma\langle Z_k\rangle)$?

8. Let $f(s_0) < \infty$ for some $s_0 > 1$. Let $G(s)$ be the function defined by the invariance $G(f(s)) = mG(s)$ for all $1 \leq s \leq f(s_0)$. Prove that

(a) $g_k(s) \searrow 1$ for $1 \leq s \leq s_0$.

(b) $G(s)$ is well-defined and unique.

(c) $0 < G(s) < \infty$ for all $1 < s \leq f(s_0)$.

(d) $G(1) = 0$ and $G'(0) = 1$ and G is continuous.

(e) $m^k(g_k(s) - 1) \searrow G(s)$ as $k \to \infty$.

9. Let Z_k be a Galton-Watson process with non-trivial offspring distribution and $m = \mathbb{E}(X) > 1$. Assume that there exists $t_0 > 0$ such that $\mathbb{E}(e^{t_0 X}) < \infty$. Let $C > 0$ and $\epsilon > 0$. Prove that there exists $\alpha > 0$ and $D > 0$ such that

$$\mathbb{P}(Z_k \geq Cm^{(1+\epsilon)k}) \leq De^{-\alpha m^{\epsilon k}}.$$

10. (Borel-Cantelli for trees) Let E_k be any measurable event for a Galton-Watson tree and write $P_k = \mathbb{P}(E_k)$. Let $\widetilde{E}$ be the event that there are infinitely many $k \in \mathbb{N}$ such that a Galton-Watson tree contains a subtree $T(v) \in E)k$ at level k. Show that,

(a) $\mathbb{P}(\widetilde{E}) = 0$ if $\sum_n P_n m^n < \infty$

(b) $\mathbb{P}(\widetilde{E}) = 1$, conditioned on non-extinction, if there exists a summable sequence K_n of non-negative numbers such that $\sum_n K_n P_n m^n = \infty$.