§1 The Galton-Watson Process and Galton-Watson Trees.

Def$^{\underline{n}}$ Let Λ be a countable index set of <u>letters</u>. Let $\Lambda^* = \bigcup_{k\in\mathbb{N}_1} \Lambda^k$ be the set of all words of finite length with letters in Λ. Write $\Lambda_0^* = \{\emptyset\} \cup \Lambda^* = \bigcup_{k\in\mathbb{N}_0} \Lambda^k$ for the set of finite words including the <u>empty word</u> $\emptyset$.

All infinite words are denoted by $\Lambda^{\mathbb{N}}$.

We write $\omega\cdot\lambda = \omega\lambda$ for the concatenation of finite and infinite words. Note:

- $\emptyset \cdot \omega = \omega \cdot \emptyset = \omega \quad \forall \omega \in \Lambda_0^*$
- $\omega\cdot\lambda \in \Lambda_0^*$ if $\omega,\lambda \in \Lambda_0^*$
- $\omega\cdot\lambda \in \Lambda^{\mathbb{N}}$ for all $\omega \in \Lambda_0^*$ and $\lambda \in \Lambda^{\mathbb{N}}$.
- $\lambda\omega$ is <u>not defined</u> for $\lambda \in \Lambda^{\mathbb{N}}$ and $\omega \in \Lambda_0^* \cup \Lambda^{\mathbb{N}}$

We write $\lambda|_n = \lambda_1\lambda_2\ldots\lambda_n \in \Lambda^n$ for all $\lambda \in \bigcup_{k=n}^{\infty} \Lambda^k \cup \Lambda^{\mathbb{N}}$.

Def$^{\underline{n}}$ A <u>rooted tree</u> over Λ is a subset $\tau \subseteq \Lambda_0^*$ s.t. $\emptyset \in \tau$ and $\forall v \in \tau \setminus \{\emptyset\}$ $\exists \lambda \in \tau$ $\exists w \in \Lambda$ with $v = \lambda w$.

That is, every node $v \in \tau$ has a unique ancestor λ.

An <u>infinite tree</u> with <u>no extinctions</u> further satisfies

$$\forall v \in \tau \quad \exists w \in \Lambda \quad \text{s.t.} \quad vw \in \tau.$$

Def$^{\underline{n}}$ The <u>Gromov boundary</u> $\partial\tau$ of τ is the subset $\partial\tau \subseteq \Lambda^{\mathbb{N}}$ s.t. $\forall v \in \partial\tau$ $\forall n \in \mathbb{N}_0$ we have $v|_n \in \tau$.

We now define the Galton-Watson tree.

Def$^{\underline{n}}$ The <u>offspring distribution</u> $\vec{\Theta} = (\Theta_0, \Theta_1, \Theta_2, \ldots)$ is a probability vector, i.e. $\sum_{k=0}^{\infty} \Theta_k = 1$.

If $\Theta_0 + \Theta_1 < 1$ we say $\vec{\Theta}$ is non-trivial.

We set $\Lambda = \mathbb{N}_1$ to address the trees.

Let X be the random variable taking values in $\mathbb{N}_0$ with probability $\vec{\Theta}$, i.e. $\mathbb{P}(X = k) = \Theta_k$.

Let $v \in \Lambda_0^*$ be a vertex in the full tree indexed by Λ_0^*. We let X_v be an i.i.d. r.v. with distribution as X (we say they are indep. copies)

For each realisation we construct the random GW tree τ thusly:

Let $L_0 = \{\emptyset\}$ and $T_0 = L_0$.

Given a set of n-length words L_n, we let

$$L_{n+1} = \bigcup_{v \in L_n} \{vw \in \Lambda^{n+1} : w \in \{1, 2, .., X_v\}\}$$

where we interpret $\{vw : w \in \{1, .., X_v\}\} = \emptyset$ if $X_v = 0$.

Further, let $T_{n+1} = T_n \cup L_{n+1}$.

This completes the induction.

We let $\tau = \lim T_n = \bigcup_{n=1}^{\infty} T_n$ be the GW tree for this specific random realisations of $(X_v)_{v \in \Lambda_0^*}$.

The nodes L_n of length n represent the "survivors" up to generation n.

The number of survivors $Z_n = \# L_n$ satisfies

$Z_{n+1} = \sum_{v \in L_n} X_v$. Hence, in distribution

$$Z_{n+1} \equiv \sum_{i=1}^{Z_n} X_{0^n i} \equiv \sum_{i=1}^{Z_n} X(i,n) \qquad (*)$$

where the X in $(*)$ are considered independent copies of X for each i, n.

We call Z_n a Galton-Watson process.

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§ 2 Probability generating functions.

PGF tell us many things encoded in real functions.

Def^{n} - The prob gen funct (PGF) $f(X,s)$ of an discrete non-neg. r.v. X is

$$\left(f_X(s) := \right) f(X,s) = \sum_{k=0}^{\infty} \mathbb{P}(X=k)\, s^k$$

(We will often omit X when clear from context)

<u>Example</u>: 1) $\vec{\theta} = (\theta_0, 1-\theta_0, 0, ..)$ gives

$$f_X(s) = \theta_0 s^0 + (1-\theta_0) s^1 = (1-\theta_0) s + \theta_0$$

E2) $\vec{\theta}(\theta_0, \theta_1, 1-\theta_0-\theta_1, 0, ..)$

$$f_X(s) = \theta_0 s^0 + \theta_1 s^1 + (1-\theta_0-\theta_1) s^2$$

$$= (1-\theta_0-\theta_1) s^2 + \theta_1 s + \theta_0$$

Note that $f(0) = \theta_0$ & $f(1) = 1$ in both cases. While the trivial case gives a linear function the quadratic is convex.

<u>Theorem</u> Let $\vec{\theta}$ be a non-trivial offspring distribution. The following hold:

1) $f(0) = \theta_0$ & $\lim_{s \searrow 0} f(s) = 0$ $\lim_{s \nearrow 1} f(s) = 1$

2) $f(s)$ is smooth on $[0,1)$ and $f^{(k)}(s)$ is cont. at $s=1$ if $f^{(k)}(1) < \infty$.

3) The offspring distribution can be recoverd from

$$\mathbb{P}(X=k) = \frac{f^{(k)}(0)}{k!}$$

4) The mean is $m = \mathbb{E}(X) = f^{(1)}(1)$

5) Assume $\mathbb{E}(X)$ is finite, then $Var(X) = \mathbb{E}((X-m)^2)$ is $Var(X) = f^{(2)}(1) + m - m^2$ [non triviality assumption]

6) $f(s)$ is strictly convex & increasing on $[0,1]$

7) If $m \leq 1$ then $f(t) > t \quad \forall t \in [0,1)$.

If $m > 1$ $\exists$ unique $q \in [0,1)$ s.t. $f(q) = q$

8) If $t \in [0,q)$ then $\underbrace{f \circ \dots \circ f}_{n \text{ times}}(t) \nearrow q$ as $n \to \infty$

If $t \in (q,1)$ then $\underbrace{f \circ \dots \circ f}_{n \text{ times}}(t) \searrow q$ as $n \to \infty$.

Proof:

1) First claim trivial,

third follows from Abel's theorem

Theorem (Abel) Let

$$g(x) = \sum_{k=0}^{\infty} a_k x^k$$

with radius of convergence 1.

Suppose $S = \sum_{k=0}^{\infty} a_k < \infty$ then $g(x) \Rightarrow S$ for $x \to 1$.

2) We compute for $s \in [0,1)$ [radius of conv. is 1]

$$f^{(1)}(s) = \sum_{k=1}^{\infty} \theta_k \cdot k \cdot s^{k-1} = \sum_{k=0}^{\infty} \theta_{k+1} (k+1) s^k$$

$$f^{(2)}(s) = \sum_{k=1}^{\infty} \theta_{k+1} (k+1) k \, s^{k-1} = \sum_{k=0}^{\infty} \theta_{k+2} (k+1)(k+2) \, s^k$$

$$\vdots$$

$$f^{(n)}(s) = \sum_{k=0}^{\infty} \theta_{k+n} (k+1)(k+2)..(k+n) \, s^k$$

Since $\vec{\theta}$ is a probability vector, $f^{(n)}(s)$ defined on $s \in [0,1)$ for all $n \Rightarrow f$ smooth on $[0,1)$.

Further, if $f^{(n)}(1) < \infty$ we have $f^{(k)}(1) < \infty$ $\forall k \leq n$ (Exercise) and by Abel's theorem we nec. have $f^{(k)}(s) \nearrow f^{(k)}(1)$.

3) Setting $f^{(k)}(0) = \theta_k \cdot (1 \cdot 2 \cdot .. \cdot k) = \theta_k \cdot k!$

$$\Rightarrow \theta_k = \frac{f^{(k)}(0)}{k!}$$

4) $$f^{(1)}(s)\Big|_{s=1} = \sum_{k=0}^{\infty} \theta_k \, k \, s^k \Big|_{s=1} = \sum_{k=0}^{\infty} \theta_k \cdot k = \mathbb{E}(X)$$

5) Exercise : Compute $Var(X) = E((X-m)^2)$.

6) Pos. constants and non-triviality give
$f^{(1)}(s) > 0$, $f^{(2)}(s) > 0$ for $s \in (0,1)$,
hence convex and str. increasing.

7) $f(0) = \theta_0$ and $f(1) = 1$, hence by convexity $f(s)$ and s can intersect at most twice on $[0,1]$. At $s = 1$, $m = f^{(1)}(1) > 1$ and so $f(1-\varepsilon) < 1-\varepsilon$ for sufficiently small $\varepsilon > 0$. Hence $\exists!$ intersection in $[0,1)$.

Ex: Proof case $m < 1$ (Hint: $m < 1 \Rightarrow \theta_0 > 0$)

8) We may assume $q > 0$ for first claim.
Then $0 \leq t < q$ and since f is str. incr.
$t < f(t) < f(q) = q \Rightarrow t < f(t) < f_2(t) < q$
$\Rightarrow \ldots \Rightarrow f_n(t) < q$.

By MCT $f_n(t)$ converges to some L s.t. $f(L) = L$ but q is least root. $\square$

The case for $t > q$ is analogous. $\square$

There is a strong connection between Z_k and the pgf of X.

Consider $f(Z_1, s) = f\left(\sum_{i=1}^{Z_0} X_{i,0}, s\right)$

$$= \mathbb{E}\left(s^{\sum_{i=1}^{Z_0} X_i}\right) = \mathbb{E}\left(\mathbb{E}\left(s^{X_1} s^{X_2} \cdots s^{X_{Z_0}} \mid \sigma\langle Z_0\rangle\right)\right)$$

$$= \mathbb{E}\left(\mathbb{E}(s^X)^{Z_0}\right) = \mathbb{E}\Big(\underbrace{\mathbb{E}(s^X)}_{f(X,s)}{}^{X'}\Big) = f(X, f(X,s)).$$

i.e. $f_{Z_1}(s) = f_X \circ f_X(s)$.

We will write $f_n = f \circ \ldots \circ f$ for the n-fold composition and $f_n^{(k)} = \frac{\partial^k}{\partial s^k}\left(f_n(\cdot, s)\right)$. (i.e. iterate first)

The observation above extends in the natural way to $f(Z_n, s) = f_n(X, s)$.

We are "hiding" the conditional expectation in the composition. This allows for simpler proofs:

Th$\underline{\text{m}}$ Let Z_k be a GW process with offspring mean $m = \mathbb{E}(X) = \mathbb{E}(Z_0) < \infty$. Then $\mathbb{E}(Z_k) = m^k$.

Pf: The chain rule gives

$$f_1^{(1)}(Z_k, s) = f_k^{(1)}(X, s) = f_{k-1}^{(1)}(X, f_1(X,s)) \cdot f_1^{(1)}(X, s)$$

$$= f_{k-2}^{(1)}\left(X, f_2(X,s)\right) \cdot f_1^{(1)}(X, f_1(X,s)) \cdot f_1^{(1)}(X,s)$$

$$\vdots$$

$$= f_1^{(1)}(X, f_{k-1}(X,s)) \cdot f_1^{(1)}(X, f_{k-2}(X,s)) \cdots f_1^{(1)}(X,s)$$

But then, $\mathbb{E}(Z_k) = f_1^{(1)}(Z_k, 1) = f_1^{(1)}(X, f_{k-1}(X,1)) \cdots f_1^{(1)}(X,1)$

$$= m^k \qquad \square$$

The critical value for which $f(q) = q$ also is the extinction probability of Z_k.

<u>Theorem</u> Let Z_k be a GW process with $\mathbb{E}(X) > 1$. Let $q \in [0,1]$ be the unique value s.t. $f(X,q) = q$. Then the probability of extinction is q,

$$\mathbb{P}(Z_k = 0 \mid \text{for large enough } k) = q.$$

<u>Proof:</u> Note that $\mathbb{P}(Z_k = 0 \mid \text{for large enough } k)$

$$= \mathbb{P}(Z_k = 0 \mid \text{for some } k) = \lim_{k \to \infty} \mathbb{P}(Z_k = 0) = \lim_k f(Z_k, 0)$$

$$= \lim_k f_k(X, 0) = q. \qquad \square$$

Example: Mandelbrot percolation.

Mandelbrot percolation or fractal percolation is a geometric random process in $\mathbb{R}^d$. Let $n \geq 2$, $p \in (0,1)$. Let $Q_0 = \{[0,1]^d\}$ be the set containing the unit cube in $\mathbb{R}^d$.

We iteratively define Q_k from Q_{k-1}:
Let $\tilde{Q}_k$ be the collection of all $n^d \cdot \# Q_{k-1}$ regular subcubes of sidelength $\left(\frac{1}{n}\right)^k$. We independently keep subcubes from $\tilde{Q}_k$ in Q_k with probability p.

Specific example, $d=2, n=2$

$Q_0 = \{\text{unit square}\}$

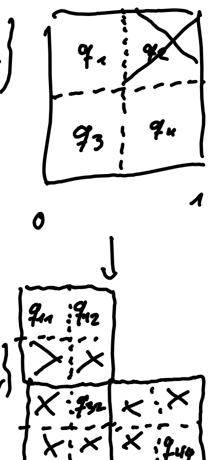
$\tilde{Q}_1 = \{q_1, q_2, q_3, q_4\}$

$Q_1 = \{q_1, q_3, q_4\}$

$\tilde{Q}_2 = \{q_{11}, q_{12}, \dots, q_{44}\}$

$Q_2 = \{q_{11}, q_{12}, q_{32}, q_{44}\}$

$\vdots$



We write $M_k = \bigcup Q_k$ for the k-generation set. The limit percolation set is given by

$$M = \lim_k M_k = \bigcap_{k=1}^{\infty} M_k .$$

We ask: When is M non-empty?

Let $\tilde{Y}$ be the retention random variable:

$$\mathbb{P}(\tilde{Y} = 1) = p, \quad \mathbb{P}(\tilde{Y} = 0) = 1-p.$$

The number of surviving subcubes of any previous generation subcube is $Y = \sum_{i=1}^{n^d} \tilde{Y}_i$, where $\tilde{Y}_i$ are iid realisations of $\tilde{Y}$.

We have $f(\tilde{Y}, s) = \sum_{k=0}^{\infty} \mathbb{P}(\tilde{Y} = k) s^k = 1-p+ps$

and $f(Y, s) = f\left(\sum^{n^d} \tilde{Y}_i, s\right) = \mathbb{E}\left(s^{\sum^{n^d} \tilde{Y}_i}\right) = \mathbb{E}\left(s^{\tilde{Y}}\right)^{n^d}$

$= f(\tilde{Y}, s)^{n^d} = (1-p+ps)^{n^d}.$

Exercise: a) Show that $\# Q_k$ is a GW process with offspring distribution $\mathrm{Bin}(n^d, p)$.

b) Let $n = d = 2$ and $p = 1/2$. Compute the probability that M is not the empty set.

c) What is the significance of $p = n^{-d}$?

d) In general, state when M is non-empty.

Application to Galton-Watson processes.

Def$^{\underline{n}}$ Let (Z_n) be a GW process with non-trivial offspring distribution $\vec{\theta}$ and mean $m = \mathbb{E} Z_0 = \mathbb{E} X < \infty$.

The <u>normalised GW-process</u> is

$$W_k = Z_k / m^k.$$

Note that $\mathbb{E}(W_k) = 1$ for all k.

<u>Lemma</u> The process (W_k) is a non-negative martingale and thus converges to some r.v. W pointwise with finite expectation.

<u>Proof</u>: Clearly $W_n \geq 0$. We check:

$$\mathbb{E}(W_{n+1} \mid \sigma\langle W_1, \dots, W_n \rangle) = \mathbb{E}\left(\frac{Z_{n+1}}{m^{n+1}} \mid \sigma\langle Z_1, \dots, Z_{n-1} \rangle \right)$$

$$= \mathbb{E}\left(\frac{1}{m^{n+1}} \sum_{i=1}^{Z_n} X_{i,n} \mid \sigma\langle Z_1, \dots Z_n \rangle \right)$$

$$= \frac{1}{m^{n+1}} \sum_{i=1}^{Z_n} \mathbb{E}(X_{i,n} \mid \sigma\langle Z_1, \dots Z_n \rangle) = \frac{1}{m^{n+1}} Z_n \cdot \mathbb{E}(X)$$

$$= \frac{Z_n}{m^n} \cdot \frac{\mathbb{E}(X)}{m} = \frac{Z_n}{m^n} = W_n \qquad \square$$

We can further conclude that $\exists C_\omega$ s.t.

$Z_k \sim C_\omega m^k$ for large k.

But C_ω may be zero for many / almost every $\omega \in \Omega$.

We want to show that W_n is in L^p for $p > 1$. Often it is easiest to check $p = 2$ since

$$\mathbb{E}(Y_n^2) \leq \mathbb{E}(Y_0^2) + \sum_{i=1}^{n} \mathbb{E}((Y_i - Y_{i-1})^2)$$

for any supermartingale (Y_n).

For GW processes there is a directer route.

Let Y have distribution $\vec{\theta}$.

Observe that $f^{(1)}(Y, s) = \sum_{i=1}^{\infty} \theta_i \cdot i \cdot s^{i-1} = \sum_{i=0}^{\infty} \theta_i \cdot i \cdot s^{i-1}$ $(s \neq 0)$

$$f^{(2)}(Y, s) = \sum_{i=2}^{\infty} \theta_i \cdot (i)(i-1) s^{i-2} = \sum_{i=0}^{\infty} \theta_i \, i(i-1) s^{i-2} \quad (s \neq 0)$$

$$= \sum_{i=0}^{\infty} \theta_i \, i^2 s^{i-2} - \sum_{i=0}^{\infty} \theta_i \, i \, s^{i-2} \implies f^{(2)}(Y, 1) = \sum_{i=0}^{\infty} \theta_i \, i^2 - f^{(1)}(Y, 1)$$

So for any non-negative integer valued r.v. Y:

$$\sum_{i=0}^{\infty} \mathbb{P}(Y=i)\, i^2 = f^{(2)}(Y,1) + f^{(1)}(Y,1) = f^{(2)}(Y,1) + \mathbb{E}(Y)$$

So, $\mathbb{E}\left((Z_k - m^k)^2\right) = \sum_{i=0}^{\infty} \mathbb{P}(Z_k = i)(i - m^k)^2$

$$= \sum_{i=0}^{\infty} \mathbb{P}(Z_k = i)\, i^2 + \sum_{i=0}^{\infty} \mathbb{P}(Z_k = i)\, m^{2k} - 2\sum_{i=0}^{\infty} \mathbb{P}(Z_k = i)\, i\, m^k$$

$$= \sum_{i=0}^{\infty} \mathbb{P}(Z_k = i)\, i^2 + m^{2k} - 2m^k \overbrace{\mathbb{E}(Z_k)}^{=m^k}$$

$$= f^{(2)}(Z_k, 1) + \mathbb{E}(Z_k) - m^{2k} = f_k^{(2)}(X,1) + m^k - m^{2k} \quad (*)$$

Now $\frac{\partial^2}{\partial s^2} f_k(X,s) = \frac{\partial}{\partial s}\left(f_{k-1}^{(1)}(X, f(X,s)) \cdot f^{(1)}(X,s)\right)$

$$= f^{(1)}(X,s)^2\, f_{k-1}^{(2)}(X, f(X,s)) + f_{k-1}^{(1)}(X, f(X,s))\, f^{(2)}(X,s) \quad \text{So,}$$

$$(*) = f_{k-1}^{(2)}(X, f(X,1))\, m^2 + m^{k-1} f^{(2)}(X,1) + m^k - m^{2k}$$

$$\vdots$$

$$= f^{(2)}(X,1)\left(m^{k-1} + m^k + m^{k+1} + \dots + m^{2k-2}\right) + m^k - m^{2k}$$

$$\leq f^{(2)}(X,1) \cdot m^{2k} + m^k - m^{2k} = \left(\mathrm{Var}(X) + m^2 - m\right) \cdot m^{2k} + m^k - m^{2k}$$

So $\mathrm{Var}(W_k) = \mathrm{Var}\left(\frac{Z_k}{m^k}\right) = \frac{1}{m^{2k}} \mathrm{Var}(Z_k) \leq \mathrm{Var}(X) + m^2 - m + m^{-k} - 1.$

and W_k is an $\mathcal{L}^2$ bounded martingale if $\mathrm{Var}(X) < \infty$.

§4 Finer Information on GW Processes.

We previously established that a normalised GW-process W_n with offspring distribution $\vec{\theta}$ (and r.v. X, Z_n) converges in L^2 if $\mathrm{Var}(X) = \mathbb{E}((X-m)^2) < \infty$.
This is equivalent to the p.g.f. f satisfying $f^{(2)}(1) < \infty$. In this section we will assume $f(s_0) < \infty$ for some $s_0 > 1$.

<u>Exercise:</u> What is the relationship between the conditions:

- $f(s_0) < \infty$ for some $s_0 > 1$
- $f^{(2)}(1) < \infty$
- $f^{(1)}(1) < \infty$
- $f(s) < \infty$ on $[0, s_0]$

Recall that for non-trivial $\vec{\theta}$ the pgf f is continuous, strictly increasing & convex on $[0, s_0]$. (assuming $f(s_0) < \infty$)
Hence there exist a strictly increasing, concave continuous inverse g on $[0, s_0]$. (That is $f \circ g = g \circ f = \mathrm{id}$).

Write $g_k = \underbrace{g \circ g \circ \cdots \circ g}_{k \text{ times}}$. Note that f is defined on $[0, s_0]$ and so g is defined on $[f(0), f(s_0)] = [\theta_0, f(s_0)]$.

We establish this technical lemma.

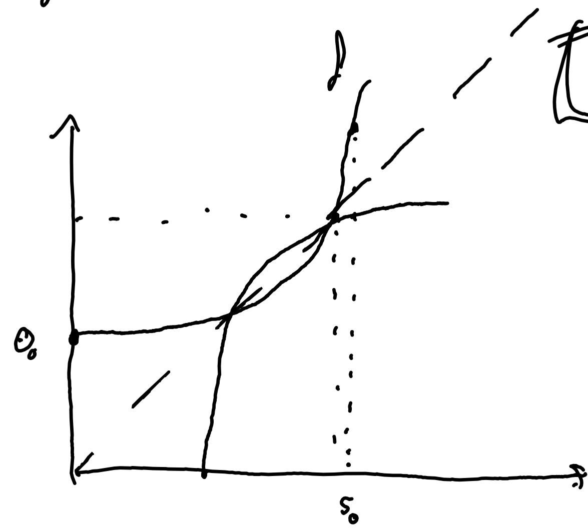
<u>Lemma</u> Let $f(s_0) < 1$ for some $s_0 > 1$. Assume $m = f^{(1)}(1) > 1$. Then, for $1 \le s \le f(s_0)$, $g_k(s) \searrow 1$

and $m^k (g_k(s) - 1) \searrow G(s)$ in $k \to \infty$,

where $G(s)$ is the unique solution to $G(f(s)) = m\,G(s)$ for $1 \le s \le f(s_0)$ which satisfies $0 < G(s) < \infty$ $\forall 1 < s \le f(s_0)$ and $G(1) = 0$ and $G'(1) = 1$, and G is continuous.

<u>Proof</u>: For $s = 1$, $G(f(1)) = G(1) = m\,G(1)$ is satisfied only by $G(1) = 0$. f is continuous with strictly increasing

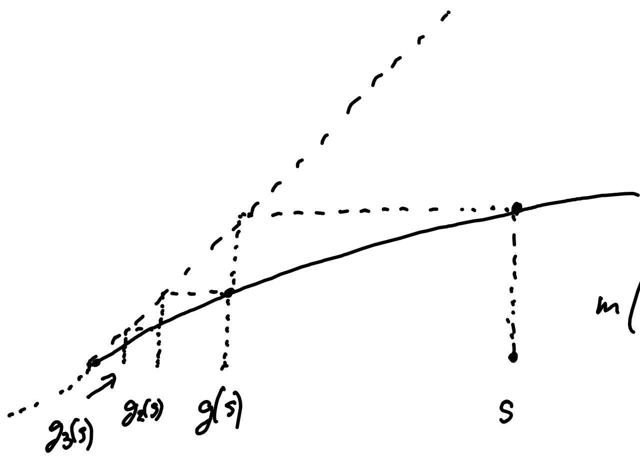
$\boxed{\text{Exercise}}$



derivative, being m at $s=1$.

$$\frac{d}{ds} G(f(s)) = G'(f(s)) f'(s) = \frac{d}{ds} m G(s) = m G'(s)$$

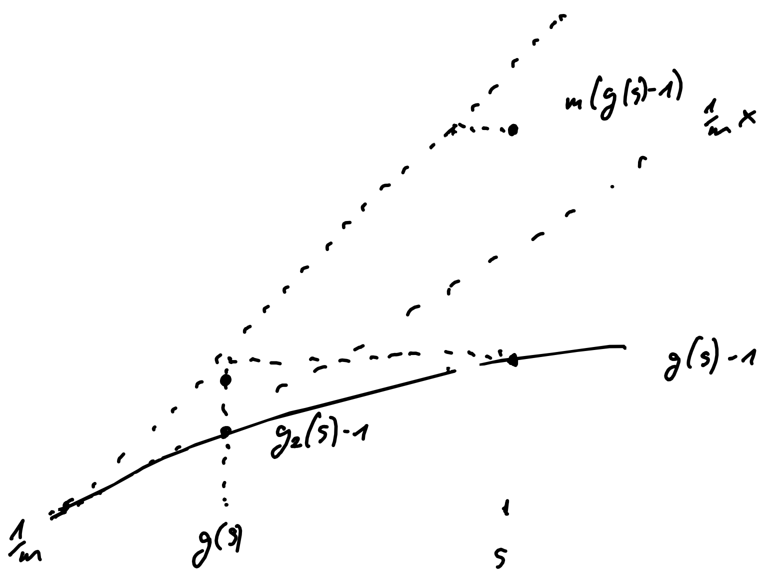
but $f'(1) = m$, so lim



$m(g(m g(s)-1)-1)$

$\vdots$

$$G(f(s)) = m\, G(s) \iff G(s) = m\, G(g(s))$$



Lemma (Athreya) Let Z_k be a GW process with mean offspring $m = \mathbb{E}(X) < \infty$. Assume there exists $t_0 > 0$ such that $\mathbb{E}(e^{t_0 W}) < \infty$. Then there exists $t_1 > 0$ s.t.

$$\sup_{k \in \mathbb{N}} \mathbb{E}(e^{t_1 W_k}) < \infty.$$

Proof: By assumption $\exists\, s_0 = e^{t_0} > 1$ s.t. $f(s_0) < \infty$. Let $K = f(s_0)$.

We must have $f_2(s) \leq K$ if $0 \leq f(s) \leq s_0$.

Equivalently, $0 \leq s \leq g(s_0)$ and in general

$$f_n(s) \leq K \quad \text{if} \quad 0 \leq s \leq g_{n-1}(s_0).$$

Note that $\mathbb{E}(e^{tW_k}) = f_k(e^{tm^{-k}})$ and so

$$\mathbb{E}(e^{tW_n}) \leq K \quad \text{if} \quad 0 \leq e^{tm^{-n}} \leq g_{n-1}(s_0)$$

$\iff t \leq m^n \log g_{n-1}(s_0)$. Now $g_{n-1}(s_0) \searrow 1$ so for large n, $\log g_{n-1}(s_0) \sim g_{n-1}(s_0) - 1$.

So using the previous lemma,

$$m_n \log g_{n-1}(s_0) \to m\, G(s_0) \quad \text{which is pos \& finite.}$$

Hence we can choose $t_1 = \inf_k m^k \log g_{k-1}(s_0)$ $\square$

$\underline{\text{Th}^{\text{m}}}$ Let Z_k be a GW process with non-trivial offspring distribution. Assume $\exists t_0 > 0$ s.t. $\mathbb{E}(e^{t_0 X}) < \infty$. Let $C > 0$ and $\varepsilon > 0$. There exists $t_2 > 0$ and $D > 0$ s.t.

$$\mathbb{P}(Z_k > C m^{(1+\varepsilon)k}) \leq D e^{-t_2 m^{\varepsilon k}}.$$

$\underline{\text{Th}^{\text{m}}}$ Let $Var(X) < \infty$ and $m = \mathbb{E}(X) > 1$.

Then: 1) $\lim_{n \to \infty} \mathbb{E}((W_n - W)^2) = 0$

already proved

2) $\mathbb{E} W = 1$, $Var\, W = \frac{Var(X)}{m^2 - m}$

3) $\mathbb{P}(W = 0) = q$

Proof: (3) Since $\mathbb{E} W = 1$, $q < 1$. By law of total probabilities

$$q_0 := \mathbb{P}(W = 0) = \sum_{k=0}^{\infty} \mathbb{P}(X_0 = k) \mathbb{P}(W_1 = 0 \mid (X_0 = k))$$

$$= \sum_{k=0}^{\infty} \theta_k \, \mathbb{P}(W = 0)^k = \sum_{k=0}^{\infty} \theta_k \, q_0^k$$

i.e. $q_0 = f(q_0)$ but q is the only solution < 1.

Hence (3) follows. $\square$