Lecture 11:

Recall: A martingale is a stochastic process $X_0, X_1, X_2, ..$ with respect to a filtration $\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2, ..$ if X_n is adapted to $\tilde{\mathcal{F}}_n$ and $\mathbb{E}(X_n \mid \tilde{\mathcal{F}}_{n-1}) = X_{n-1}$.

[Supermartingale $\leftrightarrow$ $\leq$
Submartingale $\leftrightarrow$ $\geq$]

Usually we take $\tilde{\mathcal{F}}_n = \sigma(X_1, X_2, .., X_n)$ but not always!

Remark: Let $m < n$. For every martingale, we have

$$\mathbb{E}(X_n \mid \tilde{\mathcal{F}}_m) = \mathbb{E}(\mathbb{E}(\dots \mathbb{E}(\underbrace{\mathbb{E}(X_n \mid \tilde{\mathcal{F}}_{n-1})}_{= X_{n-1}} \mid \tilde{\mathcal{F}}_{n-2}) \dots \mid \tilde{\mathcal{F}}_{m+1}) \mid \tilde{\mathcal{F}}_m)$$

$$= \mathbb{E}(\mathbb{E}(.. \underbrace{\mathbb{E}(X_{n-1} \mid \tilde{\mathcal{F}}_{n-2})}_{X_{n-2}} .. \mid \mathcal{F}_{m+1}) \mid \mathcal{F}_m)$$

$$\vdots$$

$$= \mathbb{E}(X_{m+1} \mid \tilde{\mathcal{F}}_m) = X_m.$$

<u>Definition</u> A pre-visible process is a sequence $C_1, C_2, \ldots$ of random variables, such that C_n is $\mathcal{F}_{n-1}$-measurable for all n.

Let C_n be a pre visible process. The martingale transform of X by C is

$$(C \bullet X)_n = \sum_{k=1}^{n} C_k (X_k - X_{k-1}).$$

In particular, if $C_k = 1$ for all k, then

$$(C \bullet X)_n = X_n - X_0.$$

<u>Proposition</u>: If C is a bounded pre-visible process with $|C_n(\omega)| \leq K$ for all n and $\omega \in \Omega$, then $(C \bullet X)_n$ is a martingale if X_n is.
If C is also non-negative, then $(C \bullet X)_n$ is a sub-/super martingale whenever X_n is.

Proof: We have

$$\mathbb{E}\left((C \bullet X)_n - (C \bullet X)_{n-1} \mid \mathcal{F}_{n-1} \right)$$
$$= \mathbb{E}\left(C_n (X_n - X_{n-1}) \mid \mathcal{F}_{n-1} \right)$$
$$= C_n \left(\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) - \mathbb{E}(X_{n-1} \mid \mathcal{F}_{n-1}) \right)$$
$$= C_n \left(\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) - X_{n-1} \right)$$

$$\begin{cases} = 0 & \text{if } X_n \text{ is a martingale} \\ \geq 0 & \text{if } C_n \geq 0 \text{ and } X_n \text{ is a submartingale} \\ \leq 0 & \text{if } C_n \geq 0 \text{ and } X_n \text{ is a supermartingale} \end{cases}$$

$\square$

Stopping times

A stopping time is a random variable T with values in $\{0, 1, 2, .., \infty\}$ and the property that $\{T \leq n\} = \{\omega \in \Omega : T(\omega) \leq n\} \in \mathcal{F}_n$ for all n. Equivalently, $\{T = n\} \in \mathcal{F}_n \ \forall n$.

This follows from $\{T \leq n\} = \{T \leq n-1\} \cup \{T = n\}$

Examples: • All constants are stopping times

• "First occurrence": for example:

$T = \min\{n : X_n = 0\}$ for an adapted process X_n

• If S, T are stopping times, then so are

$\min\{S, T\} = S \wedge T$ "either stopped"

and $\max\{S, T\} = S \vee T$ "both stopped"

• "Counting": for example, set

N_n = number of indices $k \leq n$ with $X_k = 0$

$T = \min\{n : N_n = 10\}$

Example: The following are (generally) <u>not</u> stopping times

$T = \max\{n : N_n = 0\}$ (we cannot determine whether $N_k = 0$ for $k > n$)

Also

$T = \min\{n : X_n = \sup_k X_k\}$ ($\sup_k X_k$ not measurable w.r.t. $\mathcal{F}_n$. Could be larger later)

Stopped processes:

Let X_n be an adapted process and T a stopping time with respect to a given filtration. The stopped process X^T is

$$X_n^T(\omega) = X_{n \wedge T(\omega)}(\omega) = \begin{cases} X_{T(\omega)}(\omega) & \text{if } n \geq T(\omega) \\ X_n(\omega) & \text{if } n < T(\omega) \end{cases}$$

Theorem: If X_n is a martingale / supermartingale / submartingale, then so is X_n^T. In particular, for every n, $\mathbb{E}(X_{T\wedge n}) \overset{\leq}{\underset{\geq}{=}} \mathbb{E}(X_0)$ $\overset{\text{super-}}{\underset{\text{sub-}}{}}$martingale.

Proof: Note that $C_n^T = \underbrace{I_{\{n \leq T\}} = 1 - I_{\{T \leq n-1\}}}_{\text{"not yet stopped at time } n-1\text{"}}$

is pre-visible.

We have $(C^T \bullet X)_n = \sum_{k=1}^{n} C_k^T (X_k - X_{k-1})$

$$= \sum_{k=1}^{n} I_{\{k \leq T\}} (X_k - X_{k-1}) = \sum_{k=1}^{T\wedge n} (X_k - X_{k-1})$$

$= X_{T\wedge n} - X_0$. So $X_{T\wedge n}$ is a martingale.

Since $\mathbb{E}(\mathbb{E}(X \mid \mathcal{F})) = \mathbb{E}(X)$, the second conclusion follows $\square$

So for every fixed n, $\mathbb{E}(X_{T\wedge n}) = \mathbb{E}(X_0)$.
Is it true that $\mathbb{E}(X^T) = \mathbb{E}(X_0)$?

In general, <u>NO!</u>

<u>Example:</u> Consider the martingale

$$X_0 = 1 \quad , \quad X_n = \begin{cases} 2X_{n-1} & \text{prob. } 1/2 \\ 0 & \text{prob. } 1/2 \end{cases}$$

Let $T = \min\{n : X_n = 0\}$. Clearly, $\mathbb{E}(X^T) = 0 \neq \mathbb{E}(X_0)$.

<u>Example:</u> Consider the simple random walk

$$X_0 = 0 \quad , \quad X_n = \begin{cases} X_{n-1} + 1 & \text{prob } 1/2 \\ X_{n-1} - 1 & \text{prob } 1/2 \end{cases}$$

Define $T = \min\{n : X_n = 1\}$. This is a stopping time and one can show that $T < \infty$ a.s.

Hence $\mathbb{E}(X^T) = 1 \neq \mathbb{E}(X_0)$.

However, under simple conditions $\mathbb{E}(X^T) = \mathbb{E}(X_0)$

Doob's Optional Stopping Theorem

Let T be a stopping time, and let X be a $\begin{matrix}\text{super-}\\ \\ \text{sub-}\end{matrix}$martingale. Suppose one of the following hold:

i) T is bounded (almost surely)

ii) X_n is bounded and $T < \infty$ for a.e. $\omega \in \Omega$.

iii) $\mathbb{E}(T) < \infty$ and $|X_n(\omega) - X_{n-1}(\omega)| \leq K$ for some fixed K, all n and almost every $\omega \in \Omega$.

Then,

$$\mathbb{E}(X^T) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} \mathbb{E}(X_0) \qquad \begin{matrix}\text{super-} \\ \text{martingale.} \\ \text{sub-}\end{matrix}$$

Proof: (i) If T is bounded by N (a.s.) we have $T \wedge N = T$ (a.s.) and so

$$\mathbb{E}(X^T) = \mathbb{E}(X_{T \wedge N}) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} \mathbb{E}(X_0).$$

(ii) We have $\mathbb{E}(X_{T \wedge n}) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} \mathbb{E}(X_0)$ for fixed n.

Since X is bounded we can use dominated convergence:

$$\mathbb{E}(X_0) = \lim_{n\to\infty} \mathbb{E}(X_{T\wedge n}) = \mathbb{E}(\lim_{n\to\infty} X_{T\wedge n}) = \mathbb{E}(X_T).$$

iii) We have

$$|X_{T\wedge n} - X_0| = \left|\sum_{k=1}^{T\wedge n} (X_k - X_{k-1})\right|$$

$$\leq \sum_{k=1}^{T\wedge n} |X_k - X_{k-1}| \leq K\cdot(T\wedge n) \leq KT.$$

So $\mathbb{E}(KT) = K\,\mathbb{E}(T) < \infty$ and we can apply DCT as above. $\square$

Corollary: If X_n is a nonnegative supermartingale and T an (almost surely) finite stopping time, then

$$\mathbb{E}(X^T) \leq \mathbb{E}(X_0)$$

<u>Proof</u>: By Fatou's lemma,

$$\mathbb{E}(X^T) = \mathbb{E}(\lim_{n\to\infty} X_{T\wedge n}) \quad \text{(why does it exist?)}$$

$$= \mathbb{E}(\liminf_{n\to\infty} X_{T\wedge n})$$

$$\leq \liminf_{n\to\infty} \mathbb{E}(X_{T\wedge n}) \leq \mathbb{E}(X_0)$$