

## Lecture 3

Recap:

An event is said to occur almost surely if its probability is 1.

Borel - Cantelli Lemma:

Let  $E_1, E_2, \dots$  be a seq. of events s.t.  
 $\sum_{i=1}^{\infty} P(E_i) < \infty$ . Then, almost surely, only  
finitely many occur

$$P(\limsup_{n \rightarrow \infty} E_n) = P(E_n \text{ occurs for i.m. } n) = 0$$

Example: Given a seq of (fair) coin tosses, let  $E_n$  be the event that the first  $n$  were heads.

Then,  $P(E_n) = \left(\frac{1}{2}\right)^n$  and

$$\sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 < \infty.$$

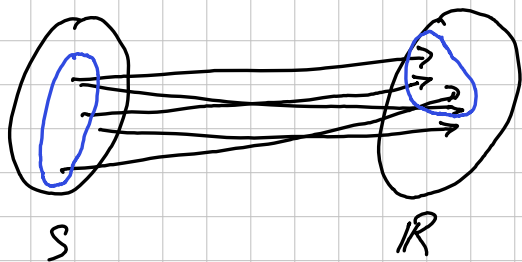
Hence the probability that  $E$  occurs i.m. times is 0, so the probability that we get just heads is 0.

# Random Variables

Def<sup>n</sup> Let  $(S, \Sigma, \mu)$  be a measure space.

We say that a function  $f: S \rightarrow \mathbb{R}$  (or  $\mathbb{R} \cup \{\pm\infty\}$ ) is **measurable** if, for all Borel sets  $A \in \sigma(\mathbb{R})$ , the pre-image satisfies

$$f^{-1}(A) = \{s \in S : f(s) \in A\} \in \Sigma.$$



We write  $m\Sigma$  for the measurable with respect to  $\Sigma$  and  $(m\Sigma)^+ \subseteq m\Sigma$  for the non-negative measurable functions and  $b\Sigma$  for the bounded measurable functions.

Remark: This can be generalised, replacing  $\mathbb{R}, \mathcal{B}(\mathbb{R})$  by any arbitrary measurable space  $T$  and  $\sigma$ -algebra  $\Sigma'$  on  $T$ .

Lemma: We have

$$1) f^{-1}(A^c) = f^{-1}(A)^c$$

$$2) f^{-1}\left(\bigcup_i A_i\right) = \bigcup f^{-1}(A_i)$$

$$3) f^{-1}\left(\bigcap_i A_i\right) = \bigcap f^{-1}(A_i)$$

Proof: We prove 2), the others are similar.

$$x \in f^{-1}\left(\bigcup_i A_i\right) \Leftrightarrow f(x) \in \bigcup_i A_i$$

$$\Leftrightarrow f(x) \in A_i \text{ for some } i$$

$$\Leftrightarrow x \in f^{-1}(A_i) \text{ for some } i$$

$$\Leftrightarrow x \in \bigcup_i f^{-1}(A_i) \quad \square$$

Prof If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous, then it is measurable w.r.t. the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$ .

This follows from the topological fact that  $f^{-1}(\mathcal{O})$  is open whenever  $\mathcal{O}$  is open and:

Proof: If  $\mathcal{C} \subseteq \mathcal{P}(\mathbb{R})$  is a collection such that

$\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$ , then  $f: S \rightarrow \mathbb{R}$  is measurable w.r.t.  $\Sigma$  if and only if  $f^{-1}(A) \in \Sigma$  for all  $A \in \mathcal{C}$ .

Proof: Let  $\mathcal{H}$  be the set of all  $A \subseteq \mathbb{R}$  such that  $f^{-1}(A) \in \Sigma$ . Clearly,  $\mathcal{C} \subseteq \mathcal{H}$ .

We show that  $\mathcal{H}$  is a  $\sigma$ -algebra.

- $f^{-1}(\mathbb{R}) = S$ , so  $\mathbb{R} \in \mathcal{H}$
- If  $A \in \mathcal{H}$ , then  $f^{-1}(A^c) = \underbrace{(f^{-1}(A))^c}_{\in \Sigma} \in \Sigma$
- If  $A_i \in \mathcal{H}$  for all  $i \in \mathbb{N}$ , then  $f^{-1}(\bigcup_i A_i) = \bigcup_i \underbrace{f^{-1}(A_i)}_{\in \Sigma} \in \Sigma$   
 $\Rightarrow \bigcup_i A_i \in \mathcal{H}$ .

Then  $\mathcal{H}$  is a  $\sigma$ -algebra that contains  $\mathcal{C}$ .

Hence  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}) \subseteq \mathcal{H}$  and

$f^{-1}(A) \in \Sigma$  for all  $A \in \mathcal{B}(\mathbb{R})$ .  $\square$

Examples: In order to show that  $f$  is measurable, it suffices to show

- $f^{-1}(A) \in \Sigma$  for all open  $A$
- $\text{---} \parallel \text{---}$  closed  $A$
- $f^{-1}((-\infty, x]) \in \Sigma$  for all  $x \in \mathbb{R}$

Lemma: If  $f_1, f_2$  are measurable functions  
 $f_1, f_2: S \rightarrow \mathbb{R}$ , then so are  $f_1 + f_2$  and  $f_1 \cdot f_2$ .

Proof: We want to show that

$(f_1 + f_2)^{-1}((x, \infty)) \in \Sigma$  for all  $x \in \mathbb{R}$ ,  
 given  $f_1^{-1}((x, \infty)), f_2^{-1}((x, \infty)) \in \Sigma$ .

We use:

$$f_1(s) + f_2(s) > x \iff \exists q \in \mathbb{Q} \text{ s.t. } f_1(s) > q, f_2(s) > x - q.$$

( $\Leftarrow$ ) is trivial: we have  $f_1(s) + f_2(s) > q + x - q = x$ .

( $\Rightarrow$ ) Let  $\varepsilon > 0$  be s.t.  $f_1(s) + f_2(s) = x + \varepsilon$ .

$$\text{i.e. } \varepsilon = f_1(s) + f_2(s) - x. \quad \begin{array}{l} f_1(s) - q < \varepsilon \\ \Downarrow \end{array}$$

Then exists  $q \in \mathbb{Q}$  s.t.  $f_1(s) - \varepsilon < q < f_1(s)$

$$\text{and so } f_2(s) = x + \varepsilon - f_1(s) > x + f_1(s) - q - f_1(s) = x - q$$

Using the fact gives  $\varepsilon \in \mathbb{Z}$

$$(f_1 + f_2)^{-1}((x, \infty)) = \bigcup_{q \in \mathbb{Q}} \left( \underbrace{f_1^{-1}((q, \infty))}_{f_1(s) > q} \cap \underbrace{f_2^{-1}((x - q, \infty))}_{f_2(s) > x - q} \right) \in \Sigma.$$

The proof for products is similar.  $\square$

Lemma: The composition of measurable functions is measurable.

Proof: This follows from the fact that

$$\begin{aligned} (f_1 \circ f_2)^{-1}(A) &= \{x : f_1(f_2(x)) \in A\} \\ &= \{x : f_2(x) \in f_1^{-1}(A)\} \\ &= f_2^{-1}(f_1^{-1}(A)) \quad \square \end{aligned}$$

Lemma If  $f_n : S \rightarrow \mathbb{R}$  is measurable for every  $n \in \mathbb{N}$  then so are

$$\inf_n f_n \quad \text{and} \quad \sup_n f_n,$$
$$\liminf_{n \rightarrow \infty} f_n \quad \text{and} \quad \limsup_{n \rightarrow \infty} f_n$$

Moreover, the set

$$\{s \in S : \lim_{n \rightarrow \infty} f_n(s) \text{ exists and is finite}\}$$

is in the  $\sigma$ -algebra  $\Sigma$ .

Proof: Note that  $(\inf_n f_n)^{-1}([x, \infty))$   
 $= \{s \in S : \inf_n f_n(s) \geq x\} = \bigcap_n \{s \in S : f_n(s) \geq x\}$   
 $= \bigcap_n f_n^{-1}([x, \infty)) \in \Sigma$  since  $f_n$  measurable.

So  $\inf_n f_n$  is measurable. The proof for  $\sup_n f_n$  is similar. Note that

$$\liminf_{n \rightarrow \infty} f_n(s) = \sup_n \inf_{m \geq n} f_m(s) \text{ and so}$$

is measurable by the previous facts. Similarly,  $\limsup_{n \rightarrow \infty} f_n(s) = \inf_n \sup_{m \geq n} f_m(s)$  is measurable.

From the last statements,

$$\begin{aligned} & \{s \in S : \lim_{n \rightarrow \infty} f_n(s) \text{ exists and is finite}\} \\ &= \{s \in S : \liminf_n f_n(s) < \infty\} \cap \{s \in S : \limsup_n f_n(s) > -\infty\} \\ & \quad \cap \{s \in S : \liminf_n f_n(s) = \limsup_n f_n(s)\} \end{aligned}$$

Since  $\left\{ \begin{array}{l} \limsup_n f_n \\ \liminf_n f_n \\ \limsup_n f_n - \liminf_n f_n \end{array} \right\}$  are measurable

and the sets above are the preimages of  $\left\{ \begin{array}{l} [-\infty, \infty) \\ (-\infty, \infty] \\ \{0\} \end{array} \right\}$

they, and their intersection, are in  $\Sigma$ .  $\square$

Def: Let  $(\Omega, \tilde{\mathcal{F}}, P)$  be a probability space. A function  $X: \Omega \rightarrow \mathbb{R}$  that is  $\tilde{\mathcal{F}}$ -measurable is called a random variable.

Example Let  $\Omega = \{1, 2, 3, 4, 5, 6\}$ ,  $\tilde{\mathcal{F}} = \mathcal{P}(\Omega)$   
and  $P(A) = \frac{\#A}{6}$ . (model of die cast)

and define 
$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in \{1, 3, 5\} \\ 0 & \text{if } \omega \in \{2, 4, 6\} \end{cases}$$

$X$  is called an indicator variable (here, of odd die rolls).

$Y(\omega) = \omega$  is also a (simple) random variable.

For any random variable  $X$  on  $(\Omega, \tilde{\mathcal{F}}, P)$

we can define the  $\sigma$ -algebra generated by

$X$  to be the smallest (sub)  $\sigma$ -algebra that

makes  $X$  measurable. This is

$\sigma(\{X^{-1}(A) : A \text{ is a Borel set}\})$  and we simply

write  $\sigma(X)$  here. (i.e.  $X$  is also a r.v on  $(\Omega, \sigma(X), P)$ ).



In the example above,

$$\sigma(Y) = \mathcal{P}(\{1, 2, 3, 4, 5, 6\})$$

$$\text{but } \sigma(X) = \{ \emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \{1, 2, 3, 4, 5, 6\} \}$$

## Law and distribution function

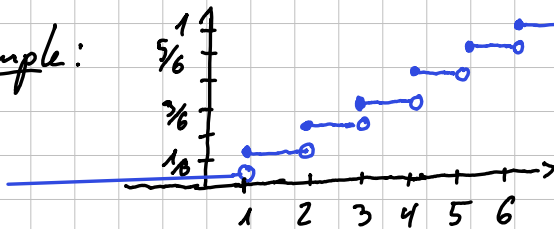
Let  $X$  be a random variable on  $(\Omega, \mathcal{F}, \mathbb{P})$ .

Define  $\mathbb{L}_X(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\})$ .  
"probability that  $X$  is in  $A$ "

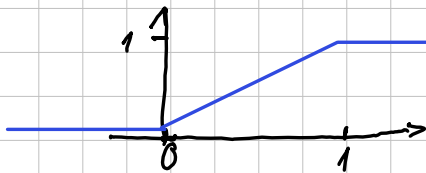
This is a probability measure on  $\mathcal{B}(\mathbb{R})$  that is called the **law of  $X$** . It is uniquely determined by the (cumulative) distribution function:

$$F_X(t) = \mathbb{L}_X((-\infty, t]) = \mathbb{P}(X \leq t)$$

Example:



Distribution of a die roll.



Distribution of a uniformly random variable on  $[0, 1]$ .

Properties of distribution functions:

- non-decreasing  $F_X(t) \leq F_X(s)$  if  $t \leq s$ .
- $\lim_{t \rightarrow -\infty} F_X(t) = 0$ ,  $\lim_{t \rightarrow \infty} F_X(t) = 1$
- right-continuous:  $\lim_{t \rightarrow a} F_X(t) = F_X(a)$ .

The third can be proven as follows:

$$\begin{aligned} \lim_{t \rightarrow a} F_X(t) &= \lim_{t \rightarrow a} \mathbb{P}(X \in (-\infty, t]) \\ &= \mathbb{P}\left(\bigcap_{t > a} \{\omega : X(\omega) \in (-\infty, t]\}\right) \\ &= \mathbb{P}(\{\omega : X(\omega) \in (-\infty, a]\}) \\ &= F_X(a). \end{aligned}$$

Conversely, given a function  $F$  satisfying all the above, we can define a probability measure  $\mathbb{P}$  with  $\mathbb{P}((-\infty, t]) = F(t)$ .