1) $$\int_{-\infty}^{\infty} \mathbb{P}(x < X \leq x+a) \, dx = \int_{-\infty}^{\infty} F(x+a) - F(x) \, dx$$

$$= \lim_{K \to \infty} \int_{-K}^{K} F(x+a) - F(x) \, dx$$

$$= \lim_{K} \left(\int_{-K}^{K} F(x+a) \, dx - \int_{-K}^{K} F(x) \, dx \right)$$

$$= \lim_{K} \left(\int_{-K+a}^{K+a} F(x) \, dx - \int_{-K}^{K} F(x) \, dx \right)$$

$$= \lim_{K} \left(\int_{K}^{K+a} F(x) \, dx - \int_{-K}^{-K+a} F(x) \, dx \right)$$

$$\leq \lim_{K} \left(a F(K+a) - F(-K+a) \right) = a$$

Approach using Fubini (covered later).

$$\int_{-\infty}^{\infty} \mathbb{P}(x < X \leq x+a) \, dx = \int_{-\infty}^{\infty} \int_{\Omega} \mathbb{1}_{[x,x+a)}(\omega) \, d\mathbb{P} \, dx$$

$$= \int_{\Omega} \int_{-\infty}^{\infty} \mathbb{1}_{[x,x+a)}(\omega) \, dx \, d\mathbb{P} = \mathbb{E}(a) = a.$$

Q2 a) $$|Y| = |\max_{1\le k\le n} X_k| \le \max_{1\le k\le n} |X_k|$$

$$\le |X_1| + |X_2| + \dots + |X_n|$$

and $\mathbb{E}|Y| \le \mathbb{E}|X_1| + \dots + \mathbb{E}|X_n| < \infty$

since n finite and $|X_j|$ integrable.

b) This follows from monotonicity of integration

$$X_k \le \max_{1\le j\le n} X_j = Y$$

and so $\mathbb{E}(X_k) \le \mathbb{E}(Y)$

c) Let $X_1 = 0$ (constant)

Let $X_2 \; -1$ (constant)

Then $\mathbb{E}|X_2| = 1$ but

$Y = \max\{X_1, X_2\} = X_1$ and

$\mathbb{E}|Y| = 0$.

Q3

"$\Rightarrow$" If Y is integrable we have

$$|X_k| \leq \max_{1 \leq j \leq n} |X_j| = |Y| \quad \text{and so}$$

$$\mathbb{E}|X_k| \leq \mathbb{E}|Y|.$$ Now let $Z = |Y|$.

"$\Leftarrow$" Assume such Z exists.

Since $Y = \max_{1 \leq j \leq n} |X_j|$ the assumption

$|X_k| \leq Z$ also implies $|Y| \leq Z$.

Since $\mathbb{E}(|Y|) \leq \mathbb{E}(|Z|) < \infty$, Y is integrable.

Q4.
$$\mathbb{E}(m_n^2) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left(\left(X_k - \bar{X}_n\right)^2\right)$$

$$= \frac{1}{n} \sum_{k=1}^{n} \left(\mathbb{E}(X_k^2) + \mathbb{E}(\bar{X}_n^2) - 2\mathbb{E}(X_k \bar{X}_n)\right)$$

$$= \mathbb{E}(X_1^2) + \mathbb{E}(\bar{X}_n^2) - 2\mathbb{E}(X_1 \bar{X}_n)$$

$$= \mathbb{E}(X_1^2) + \frac{1}{n^2} \mathbb{E}\left(\left(\sum_{k=1}^{n} X_k\right)^2\right) - 2\mathbb{E}\left(X_1 \frac{1}{n}\left(\sum_{k=1}^{n} X_k\right)\right)$$

$$= \mathbb{E}(X_1^2) + \frac{1}{n^2} \underbrace{\mathbb{E}\left(\left(\sum_{k=1}^{n} X_k\right)^2\right)}_{(*)} - 2 \underbrace{\mathbb{E}\left(X_1 \frac{1}{n}\left(\sum_{k=1}^{n} X_k\right)\right)}_{(\dagger)} \quad (\ddagger)$$

$$(*) = \mathbb{E}\left(\sum_{k=1}^{n} X_k^2 + \sum_{k=1}^{n} \sum_{i \neq k} X_k X_i\right)$$

$$= n\,\mathbb{E}(X_1^2) + n(n-1)\,\mathbb{E}(X_1)^2$$

$$(\dagger) = \frac{1}{n} \mathbb{E}\left(X_1^2 + \sum_{k=2}^{n} X_1 X_k\right) = \frac{1}{n}\mathbb{E}(X_1^2) + \frac{n-1}{n}\mathbb{E}(X_1)^2$$

$$(\ddagger) = \mathbb{E}(X_1^2) + \frac{\mathbb{E}(X_1^2)}{n} + \frac{n-1}{n}\mathbb{E}(X_1)^2 - \frac{2}{n}\mathbb{E}(X_1^2) - 2\frac{n-1}{n}\mathbb{E}(X_1)^2$$

$$= \mathbb{E}(X_1^2) - \frac{1}{n}\mathbb{E}(X_1^2) - \frac{n-1}{n}\mathbb{E}(X_1)^2$$

$$= \frac{n-1}{n}\left(\mathbb{E}(X_1^2) - \mathbb{E}(X_1)^2\right) = \frac{n-1}{n} \mathrm{Var}(X_1)$$

$\mathrm{Var}(m_n^2)$ calculation similar. Details omitted.

5) First note that

$$X > n \implies \frac{n}{X} < 1 \quad \text{whenever } X \neq 0$$

Hence

$$\mathbb{E}\left(\frac{n}{X} I_{X>n}\right) = \int \frac{n}{X} I_{X>n} \, d\mathbb{P} < \int 1 \, I_{X>n} \, d\mathbb{P}$$

$= 1 - F_X(n)$. But $F_X(n) \to 1$ as $n \to \infty$

giving the first statement.

For the second, we note that

$$\frac{X \cdot I_{X<n}}{n} \leq \frac{n}{n} = 1. \quad \text{Since } \mathbb{E}(|1|) = 1 < \infty$$

we may use the DCT to the seq.

$Y_n = \frac{X \cdot I_{X<n}}{n}$ which converges pointwise

to 0 as $Y_n = \frac{X(\omega) \cdot I(\omega)}{n} \leq \frac{X(\omega)}{n} \to 0$.

So $Y_n \to 0$ and $\lim_{n \to \infty} \mathbb{E}(Y_n) = \mathbb{E}(0) = 0$. $\square$

6.)

a) Let $t \in [-\theta, \theta]$. Then, there exists $p \in [0,1]$ s.t. $t = p(-\theta) + (1-p)\theta$

Since X is non-negative, for any fixed $X \geq 0$ $t \mapsto e^{Xt}$ is convex. Thus,

$$e^{Xt} \leq p e^{-\theta X} + (1-p) e^{\theta X}.$$

Using monotinicity,

$$\mathbb{E}(e^{Xt}) \leq p \, \mathbb{E}(e^{-\theta X}) + (1-p) \mathbb{E}(e^{\theta X})$$
$$\leq \mathbb{E}(e^{-\theta X}) + \mathbb{E}(e^{\theta X}) < \infty$$

Hence there exists such $C > 0$.

b) Let $f(t,X) : \mathbb{R} \to \mathbb{R}_0^+$ be measurable.

Assume that f is uniformly differentiable w.r.t. t on a neighbourhood $(t_0 - a, t_0 + a)$. We first show, assuming $\mathbb{E}\left(\left|\frac{\partial}{\partial t} f(t,X)\right|_{t=t_0}\right|\right) < \infty$, that

$$\frac{\partial}{\partial t} \mathbb{E}(f(t,X))\Big|_{t=t_0} = \mathbb{E}\left(\frac{\partial}{\partial t} f(t,X)\Big|_{t=t_0}\right).$$

The LHS can be expressed as

$$\lim_{n\to\infty} \frac{\mathbb{E}(f(t_0+a_n, X)) - \mathbb{E}(f(t_0, X))}{a_n},$$

where $|a_n| < a$ is any seq. such that $a_n \to 0$

This gives

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{f(t_0+a_n, X) - f(t_0, X)}{a_n} \right)$$

$$= \lim_{n\to\infty} \mathbb{E}\left(\frac{\partial}{\partial t} f(t, X)\Big|_{t=t_0} + \delta_n \right),$$ where the error

$\delta_n \to 0$ as $n \to \infty$. Since $\mathbb{E}(|\frac{\partial}{\partial t} f(t, X)|) < \infty$

this is bounded. Applying the DCT gives

$$\frac{\partial}{\partial t} \mathbb{E}(f(t, X))\Big|_{t=t_0} = \lim_{n\to\infty} \mathbb{E}\left(\frac{\partial}{\partial t} f(t, X)\Big|_{t=t_0} + \delta_n\right) = \mathbb{E}\left(\frac{\partial}{\partial t} f(t, X)\Big|_{t=t_0}\right)$$

The argument can be repeated for all t_0 in an open interval and hence we may differentiate under the expectation, assuming integrability. Hence,

$$\frac{d^k}{dt^k} M_X(t) = \mathbb{E}\left(\frac{d^k}{dt^k} e^{tX}\right) = \mathbb{E}(X^k e^{tX}),$$

provided the latter is integrable. But

$$\mathbb{E}(X^k e^{tX}) \leq \mathbb{E}(e^{kX} e^{tX}) = \mathbb{E}(e^{(t+k)X}) < \infty$$

by assumption $\square$

c) To compute the k-th moment, we consider $\frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \mathbb{E}(X^k e^{tX})\Big|_{t=0} = \mathbb{E}(X^k)$.

Hence the k-th moment is given by the k-th derivative of $M_X(t)$ at 0. In particular, under these conditions it exists.

7) Let X_n be the length of stick after the n-th breaking. $X_0 = 1$, $X_1 \sim U[0,1]$, $X_2 \sim U[0, X_1]$, $X_3 \sim U[0, X_2]$, ..

We note that we can equivalently write

$X_0 = 1$, $X_1 = Z_1$, $X_2 = Z_1 Z_2$, $X_3 = Z_1 Z_2 Z_3$,

where $Z_k \sim U[0,1]$. Then,

$$X_n = \prod_{j=1}^{n} Z_j \quad \text{and} \quad \log X_n = \sum_{j=1}^{n} \log Z_j$$

$du = 4\log^3(x) \cdot \frac{1}{x}\, dx$

$v = x$

$$\text{Now } \mathbb{E}(|\log^4 Z_j|) = \int_0^1 \underbrace{\log^4 x}_{u} \, \underbrace{dx}_{dv}$$

$$= x\log^4 x\Big|_0^1 - \int_0^1 4\log^3(x)\, dx = 0 - 4x\log^3 x\Big|_0^1 + \int_0^1 12 \log^2(x)\, dx$$

$$= \dots = 24 < \infty$$

So it has fourth moment and we can apply the LLN:

$$\lim_{n\to\infty} \frac{\log X_n}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log Z_j \to \mathbb{E}(\log Z_1)$$

$$= \int_0^1 \log x \, dx = (x\log x - x)\Big|_0^1 = -1.$$

8)

Using the hint,

$$\mathbb{E}(X) = \mathbb{E}(X I_{X\geq\lambda\mathbb{E}X} + X I_{X<\lambda\mathbb{E}X})$$

$$= \mathbb{E}(X I_{X\geq\lambda\mathbb{E}X}) + \mathbb{E}(X I_{X<\lambda\mathbb{E}X})$$

$$\leq \mathbb{E}(X I_{X\geq\lambda\mathbb{E}X}) + \lambda\mathbb{E}(X)$$

$$\Rightarrow \mathbb{E}(X)(1-\lambda) \leq \mathbb{E}(X I_{X\geq\lambda\mathbb{E}X})$$

$$\underset{C.S.}{\leq} \sqrt{\mathbb{E}(X^2)\,\mathbb{E}(I^2_{X\geq\lambda\mathbb{E}X})} = \sqrt{\mathbb{E}(X^2)}\sqrt{P(X\geq\lambda\mathbb{E}X)}$$

$$\Rightarrow \mathbb{E}(X^2)P(X\geq\lambda\mathbb{E}X) \geq (1-\lambda)^2\,\mathbb{E}(X)^2$$

$$\Rightarrow P(X\geq\lambda\mathbb{E}X) \geq (1-\lambda)^2\frac{\mathbb{E}(X)^2}{\mathbb{E}X^2}$$

(This is known as the Paley - Zygmund ineq.)

9 a)

Let $X_k = \begin{cases} 1 & \text{if dots at position } k \text{ and } k+1 \text{ are red} \\ 0 & \text{otherwise} \end{cases}$

For each $1 \leq k < n$, $\mathbb{P}(X_k > 0) = p_n^2$

Note however that X_k, X_{k+1} are not independent.

Further, $P_n = \sum_{k=1}^{n-1} X_k$.

Now $\mathbb{E}(P_n) = \mathbb{E}\left(\sum_{k=1}^{n-1} X_k\right) = \sum_{k=1}^{n-1} \mathbb{E}X_k = (n-1)p_n^2$

b)

$q_n := \mathbb{P}(P_n > 0) = \mathbb{P}(P_n \geq 1) \leq \mathbb{E}(P_n)$ (Markov)

Combining with a) gives $q_n \leq (n-1)p_n^2$

Assuming $\sqrt{n}p_n \to 0$ gives $p_n \leq \varepsilon/\sqrt{n}$ for all $\varepsilon > 0$ for large enough n, and $p_n^2 \leq \varepsilon/n$

Thus $q_n \leq \frac{(n-1)\varepsilon^2}{n} \leq \varepsilon^2$. Since $\varepsilon > 0$ was arbitrary the desired conclusion follows.

c) We assume there exists a sequence n_k and $\varepsilon > 0$ s.t. $\sqrt{n_k}\, p_{n_k} \geq \varepsilon \Rightarrow p_{n_k}^2 \geq \varepsilon^2/n_k$

As noted above X_k and X_{k+1} are not independent.

However, X_k and X_{k+2} are independent. We can thus bound $\mathbb{P}(P_{n_k} > 0)$ below by

$$P(P_{n_k}>0) \geq P\left(\sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} X_{2j} > 0\right)$$

$$= 1 - P\left(X_{2j} = 0 \ : \ 1 \leq j \leq \lfloor \tfrac{n_k - 1}{2} \rfloor\right)$$

$$= 1 - \prod_{j=1}^{\lfloor \frac{n_k-1}{2} \rfloor} P(X_{2j} = 0) \qquad \text{(independence)}$$

$$= 1 - (1 - p_{n_k}^2)^{\lfloor \frac{n_k-1}{2} \rfloor} \geq 1 - (1 - p_{n_k}^2)^{n_k}$$

$$= 1 - \exp\left(n_k \log(1 - p_{n_k}^2)\right)$$

$$\geq 1 - \exp(-n_k p_{n_k}^2) \geq 1 - \exp(-\varepsilon^2) > 0.$$

This proves our claim. $\square$

10)

a)

$$X_H = \begin{cases} 0 & \omega \in \{TTT\} \\ 1 & \omega \in \{HTT, THT, TTH\} \\ 2 & \omega \in \{THH, HTH, HHT\} \\ 3 & \omega \in \{HHH\} \end{cases}$$

b)

$$Y = \begin{cases} 0, \ \omega \in \{HTT, THT, TTH\} \\ 1, \ \omega \in \{TTT, THH, HTH, HHT\} \\ 4, \ \omega \in \{HHH\} \end{cases}$$

$$E = \{HTT, THT, TTH, HHH\}$$

We get $\sigma(E) = \{\emptyset, \Omega, E, E^c\}$

$\sigma(X) = \{\emptyset, \Omega, \{TTT\}, \{HHH\}, \{TTT, HHH\}, \ldots\}$

$\sigma(Y) = \{\emptyset, \Omega, \{HHH\}, \{HTT, THT, TTH\}, \ldots\}$

$Z = \mathbb{E}(X \mid E)$ must be constant on E and E^c by measurability. Hence $Z = \begin{cases} a & \omega \in E \\ b & \omega \in E^c \end{cases}$

and $\int_E Z \, d\mathbb{P} = \mathbb{P}(E) a = \int_E X \, d\mathbb{P} = 1 \cdot \mathbb{P}(\{HTT, THT, TTH\}) + 3 \cdot \mathbb{P}(\{HHH\})$

$\Rightarrow \quad a \frac{4}{8} = 1 \cdot \frac{3}{8} + 3 \frac{1}{8} \Rightarrow a = \frac{6}{4} = \frac{3}{2}$

Similarly $b \frac{4}{8} = 0 \cdot \frac{1}{8} + 2 \cdot \frac{3}{8} \Rightarrow b = \frac{6}{4} = \frac{3}{2}$

So $Z(\omega) = \frac{3}{2}$.

For $Z = \mathbb{E}(X \mid Y)$, we must have Z constant on $\{HHH\}$, $\{HTT, THT, TTH\}$, and $\{TTT, THH, HTH, HHT\}$

This gives

$$Z = \begin{cases} 3 & \omega \in \{HHH\} \\ 1 & \omega \in \{HTT, THT, TTH\} \\ 3/2 & \omega \in \{TTT, THH, HTH, HHT\} \end{cases}$$

For $Z = \mathbb{E}(Y \mid X)$ we get

$$Z(\omega) = \begin{cases} 1 & \omega \in \{TTT\} \\ 0 & \omega \in \{HTT, THT, TTH\} \\ 1 & \omega \in \{THH, HTH, HHT\} \\ 4 & \omega \in \{HHH\} \end{cases} = Y(\omega)$$

Since $\int_{\{TTT\}} Z = Z(TTT)\,\mathbb{P}(TTT) = \int_{\{TTT\}} Y = Y\{TTT\}\,\mathbb{P}(TTT)$

$$\int_{\{HTT, THT, TTH\}} Z = a\,\mathbb{P}(HTT, .., TTH) = \int_{\{HTT, .., TTH\}} Y = 0$$

$$\int_{\{THH, HTH, HHT\}} Z = a\,\mathbb{P}(THH, .., HHT) = \int_{\{THH, .., HHT\}} Y = 1 \cdot \mathbb{P}(THH, .., HHT)$$