Overlapping iterated function systems from the perspective of Metric Number Theory

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Given an iterated function system (IFS) $\Phi = \{\varphi_a\}_{a\in\mathcal{A}}$ there exists a unique non-empty compact set X satisfying

$$X = \bigcup_{a\in\mathcal{A}} \varphi_a(X).$$

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Given an IFS Φ and a probability vector $\mathbf{p} = (p_a)_{a\in\mathcal{A}}$, there exists a unique Borel probability measure $\mu_{\mathbf{p}}$ satisfying

$$\mu_{\mathbf{p}} = \sum_{a\in A} p_a \cdot \varphi_a \mu_{\mathbf{p}}.$$

These measures are well studied objects. They provide a lot of information about X.

For any $z \in X$ we have

$$\mu_{\mathbf{p}} = \lim_{n \to \infty} \sum_{\mathbf{a} \in \mathcal{A}^n} p_{\mathbf{a}} \cdot \delta_{\varphi_{\mathbf{a}}(z)}.$$

Where for a word $\mathbf{a} = (a_1, \ldots, a_n)$ we have

$$p_{\mathbf{a}} = \prod_{i=1}^{n} p_{a_i} \text{ and } \varphi_{\mathbf{a}} = \varphi_{a_1} \circ \cdots \circ \varphi_{a_n}.$$

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So any property of the measure $\mu_{\mathbf{p}}$ you may be interested in, e.g. dimension, absolute continuity, etc, can be viewed as information about the distribution of the set of points $\{\varphi_{\mathbf{a}}(z)\}_{\mathbf{a}\in\mathcal{A}^n}$ in the limit.

The set $\cup_{n=1}^{\infty}\{\varphi_{\mathbf{a}}(z)\}_{\mathbf{a}\in\mathcal{A}^n}$ is dense in X, and as n increases the sets $\{\varphi_{\mathbf{a}}(z)\}_{\mathbf{a}\in\mathcal{A}^n}$ become “more dense”.

This resembles how the rational numbers are distributed within $\mathbb{R}$. The study of how the rational numbers are distributed within $\mathbb{R}$ is known as Diophantine Approximation.

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Can we study iterated function systems using ideas from Diophantine Approximation?

- Do we observe analogues of classical results from Diophantine Approximation in a fractal setting?

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- Does viewing iterated function systems through this Diophantine lens provide a new classification for iterated function systems? YES
- Does this approach allow for new insights into existing work? YES

Given a function $\Psi : \mathbb{N} \to [0, \infty)$ we can define a limsup set as follows:

$$W_\Psi := \{x \in \mathbb{R} : |x - p/q| \leq \Psi(q) \text{ for i.m. } (p, q) \in \mathbb{N} \times \mathbb{Z}\}.$$

Theorem (Khintchine 1924)

The following statements are true

- *Suppose $\sum_{q=1}^{\infty} q \cdot \Psi(q) < \infty$ then W_Ψ has Lebesgue measure zero.*
- *Suppose Ψ is decreasing and $\sum_{q=1}^{\infty} q \cdot \Psi(q) = \infty$ then Lebesgue almost every x is contained in W_Ψ.*

Let $\Phi = \{\varphi_a\}_{a\in A}$ be an IFS. Given $z \in X$ and $\Psi : \cup_{n=1}^{\infty}\mathcal{A}^n \to [0, \infty)$ we define

$$W_\Phi(\Psi, z) := \{x \in \mathbb{R}^d : |x - \varphi_{\mathbf{a}}(z)| \leq \Psi(\mathbf{a}) \text{ for i.m. } \mathbf{a} \in \cup_{n=1}^{\infty}\mathcal{A}^n\}.$$

Related works that study these sets include papers by Allen and Barany, B., B. and Troscheit, Levesley, Salp and Velani, and Persson and Reeve.

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We will always assume that Φ is such that X has positive Lebesgue measure (or Φ belongs to a family for which generically X has positive Lebesgue measure). This assumption leads to more interesting behaviour.

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Suppose $\Phi_1 = \{\varphi_{1,a}\}_{a\in A}$ and $\Phi_2 = \{\varphi_{2,a}\}_{a\in A}$ are two IFSs that belong to some parameterised family (e.g. Bernoulli convolutions), if $W_{\Phi_1}(\Psi, z_1)$ has full measure and $W_{\Phi_2}(\Psi, z_2)$ has zero measure then the images of z_1 under Φ_1 are more evenly distributed throughout X_1 then the images of z_2 under Φ_2 within X_2.

If

$$\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathcal{A}^n}\Psi(\mathbf{a})^d<\infty$$

then $W_\Phi(\Psi, z)$ has zero Lebesgue measure for any $z \in X$. Does divergence imply full measure?

We restrict to Ψ of the following form

$$\Psi(\mathbf{a}) = \left(\frac{h(n)}{\#\mathcal{A}^n}\right)^{1/d} \text{ for } \mathbf{a} \in \mathcal{A}^n$$

Where $h : \mathbb{N} \to [0, \infty)$.

Notice that for these Ψ

$$\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathcal{A}^n}\Psi(\mathbf{a})^d \quad \text{simplifies to} \quad \sum_{n=1}^{\infty}h(n).$$

We will require some further restriction on h. We say that h is good if it satisfies the following properties:

- There exists $\epsilon > 0$ such that for any $B \subset \mathbb{N}$ satisfying $\overline{d}(B) > 1 - \epsilon$ we have $\sum_{n \in B} h(n) = \infty$. [1]
- There exists $c > 0$ such that $\frac{h(n+1)}{h(n)} > c$ for all $n \in \mathbb{N}$.

[1] $\overline{d}(B) = \limsup_{N \to \infty} \frac{\#\{1 \leq n \leq N : n \in B\}}{N}$

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The function given by $h(n) = \frac{1}{n}$ is good.

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If $\{\varphi_a\}_{a\in\mathcal{A}}$ has an exact overlap ($\varphi_{\mathbf{a}} = \varphi_{\mathbf{b}}$ for some $\mathbf{a} \neq \mathbf{b}$), then for any bounded h the set

$$\left\{x \in \mathbb{R}^d : |x - \varphi_{\mathbf{a}}(z)| \leq \left(\frac{h(n)}{\#\mathcal{A}^n}\right)^{1/d} \text{ for i.m. } (a_1, \ldots, a_n) \in \cup_{m=1}^{\infty}\mathcal{A}^m\right\}$$

has zero Lebesgue measure for any $z \in X$.

Theorem (B)

For Lebesgue almost every $\lambda \in (1/2, 0.668)$, for any good h Lebesgue almost every $x \in [\frac{-1}{1-\lambda}, \frac{1}{1-\lambda}]$ is contained in

$$\left\{x \in \mathbb{R} : \left|x - \sum_{i=1}^{n} a_i \lambda^{i-1}\right| \leq \frac{h(n)}{2^n} \textit{ for i.m. } (a_1, \ldots, a_n) \in \cup_{m=1}^{\infty}\{-1, 1\}^m\right\}$$

For this theorem the relevant IFS is $\{\varphi_0(x) = \lambda x - 1,\ \varphi_1(x) = \lambda x + 1\}$ where and $z = 0$.

Given a finite set of matrices $\{T_a\}_{a\in\mathcal{A}}$ each satisfying $\|T_a\| < 1$ we can define a parameterised family of IFSs by associating to each $\mathbf{t} = (t_1, \ldots, t_{\#\mathcal{A}}) \in \mathbb{R}^{\#\mathcal{A}\cdot d}$ the IFS

$$\{\varphi_a(x) = T_a x + t_a\}_{a\in\mathcal{A}}.$$

We let $X_{\mathbf{t}}$ denote the corresponding attractor and let $\pi_{\mathbf{t}} : \mathcal{A}^{\mathbb{N}} \to X_{\mathbf{t}}$ denote the projection map given by

$$\pi_{\mathbf{t}}((b_j)) = \lim_{n\to\infty} (\varphi_{b_1} \circ \cdots \circ \varphi_{b_n})(0).$$

Theorem (B)

Assume that $\|T_a\| < 1/2$ *for all* $a \in A$ *and that the Lyapunov dimension*[2] *exceeds* 1. *Let* $(b_j) \in \mathcal{A}^{\mathbb{N}}$. *Then for Lebesgue almost every* $\mathbf{t} \in \mathbb{R}^{\#\mathcal{A} \cdot d}$, *for any good* h *the set*

$$\left\{ x \in \mathbb{R}^d : |x - \varphi_{\mathbf{a}}(\pi_{\mathbf{t}}((b_j)))| \leq \left(\frac{h(n)}{\#\mathcal{A}^n}\right)^{1/d} \text{ for i.m. } \mathbf{a} = (a_1, \ldots, a_n) \right\}$$

has positive Lebesgue measure.

[2] With respect to the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$.

Theorem (B)

Assume that $\|T_a\| < 1/2$ for all $a \in A$, the Lyapunov dimension exceeds 1, and that one of the following properties holds:

- *Each φ_a is a similarity.*
- *$d = 2$ and each $T_a = T_{a'}$ for $a \neq a'$.*
- *All of the T_a are simultaneously diagonalisable.*

Let $(b_j) \in \mathcal{A}^{\mathbb{N}}$. Then for Lebesgue almost every $\mathbf{t} \in \mathbb{R}^{\#\mathcal{A}\cdot d}$, for any good h Lebesgue almost every $x \in X_{\mathbf{t}}$ is contained in

$$\left\{ x \in \mathbb{R}^d : |x - \varphi_{\mathbf{a}}(\pi_{\mathbf{t}}((b_j)))| \leq \left(\frac{h(n)}{\#\mathcal{A}^n}\right)^{1/d} \textit{ for i.m. } \mathbf{a} = (a_1, \ldots, a_n) \right\}$$

These results can be generalised to more exotic Ψ. Given a "nice" measure $\mathfrak{m}$ on $\mathcal{A}^{\mathbb{N}}$ we can define a family of Ψ given by

$$\Psi(\mathbf{a}) = (\mathfrak{m}([\mathbf{a}]) \cdot h(n))^{1/d} \text{ for } \mathbf{a} \in \mathcal{A}^n.$$

For these Ψ we have analogous results.

Theorem (B)

Let $\Phi = \{\varphi_a\}_{a\in\mathcal{A}}$ be an IFS and $z \in X$. Assume that for any $h : \mathbb{N} \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} h(n) = \infty$ the set

$$\left\{ x : |x - \varphi_{\mathbf{a}}(z)| \leq \left(\frac{h(n)}{\#\mathcal{A}^n}\right)^{1/d} \textit{ for i.m. } \mathbf{a} \in \cup_{m=1}^{\infty} \mathcal{A}^m \right\}$$

has positive Lebesgue measure. Then the $\mu_{\mathbf{p}}$ corresponding to $(\frac{1}{\#\mathcal{A}}, \ldots, \frac{1}{\#\mathcal{A}})$ is absolutely continuous.

This theorem holds for pushforwards of more general measures.

We now focus on one specific family. Given $t \in [0, 1]$ let

$$\Phi_t := \left\{ \varphi_1(x) = \frac{x}{2}, \varphi_2(x) = \frac{x+1}{2}, \varphi_3(x) = \frac{x+t}{2}, \varphi_4(x) = \frac{x+1+t}{2} \right\}.$$

For each Φ_t the self-similar set is $[0, 1+t]$.

Given $t \in [0,1]$, $h : \mathbb{N} \to [0,\infty)$, and $z \in [0,1+t]$, let $W_t(h,z)$ denote the following set

$$\left\{ x : |x - \varphi_{\mathbf{a}}(z)| \leq \frac{h(n)}{4^n} \text{ for i.m. } (a_1, \ldots, a_n) \in \cup_{m=1}^{\infty} \{1,2,3,4\}^m \right\}.$$

Theorem (B)

The following statements are true:

- *If* $t \in \mathbb{Q}$ *then* Φ_t *contains an exact overlap and* $\dim_H(W_t(1, z)) < 1$ *for any* $z \in [0, 1 + t]$.
- *If* $t \notin \mathbb{Q}$ *then there exist* h *satisfying* $\lim_{n\to\infty} h(n) = 0$, *and for any* $z \in [0, 1 + t]$ *Lebesgue almost every* x *is contained in* $W_t(h, z)$.

Theorem (B)

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- *If t is badly approximable then for any h satisfying* $\sum_{n=1}^{\infty} h(n) = \infty$, *for any $z \in [0, 1+t]$ Lebesgue almost every x is contained in $W_t(h,z)$.*
- *If t is not badly approximable then there exists h satisfying* $\sum_{n=1}^{\infty} h(n) = \infty$, *such that $W_t(h,z)$ has zero Lebesgue measure for any $z \in [0, 1+t]$.*

Combining this theorem with results of Hochman, and Shmerkin and Soloymak, it can be shown that there exists t, t' such that Φ_t and Φ'_t satisfy the following:

- $\dim_H \mu_{\mathbf{p},t} = \dim_H \mu_{\mathbf{p},t'} = \min\left\{1, \frac{h(\mathbf{p})}{\log 2}\right\}$ for any probability vector $\mathbf{p}$.
- $\{\mathbf{p} : \mu_{\mathbf{p},t} \text{ is absolutely continuous}\}$ coincides with $\{\mathbf{p} : \mu_{\mathbf{p},t'} \text{ is absolutely continuous}\}$
- There exists $h : \mathbb{N} \to [0,\infty)$ such that $W_t(h,z)$ has full measure for all z, and $W_{t'}(h,z)$ has zero measure for all z.

In other words, Φ_t and $\Phi_{t'}$ are indistinguishable in terms of the behaviour of their self-similar measures, but distinguishable when viewed from this Diophantine perspective.

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1. Given an IFS $\{\varphi_a\}_{a\in\mathcal{A}}$ and $z\in X$, we show that there exists $c_1, c_2 > 0$ such that for a large infinite set $B\subset\mathbb{N}$ we have the following:
   - If $n\in B$ there exists $S_n\subset\mathcal{A}^n$ satisfying $\#S_n\geq c_1\cdot\#\mathcal{A}^n$ with the property that if $\mathbf{a},\mathbf{b}\in S_n$ and $\mathbf{a}\neq\mathbf{b}$ then

$$|\varphi_{\mathbf{a}}(z)-\varphi_{\mathbf{b}}(z)|\geq\left(\frac{c_2}{\#\mathcal{A}^n}\right)^{1/d}.$$

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$$|\varphi_{\mathbf{a}}(z)-\varphi_{\mathbf{b}}(z)|\geq\left(\frac{c_2}{\#\mathcal{A}^n}\right)^{1/d}.$$

2. Given a good h use 1. to prove that the following set has positive Lebesgue measure

$$\bigcap_{N=1}^{\infty}\bigcup_{n\geq N:n\in B}\bigcup_{\mathbf{a}\in S_n}B\left(\varphi_{\mathbf{a}}(z),\left(\frac{h(n)}{\#\mathcal{A}^n}\right)^{1/d}\right).$$

3 Use the properties of h to improve positive measure to full measure.

The key step is 1. To establish the existence of a large well separated set we study the quantity

$$\#\left\{(\mathbf{a},\mathbf{b})\in\mathcal{A}^n : \mathbf{a}\neq\mathbf{b},\ |\varphi_{\mathbf{a}}(z)-\varphi_{\mathbf{b}}(z)|\leq\left(\frac{s}{\#\mathcal{A}^n}\right)^{1/d}\right\}.$$

For parameterised families of IFSs this quantity can be studied using the transversality technique (Benjamini and Solomyak). For the family Φ_t it can be studied using the Diophantine properties of t.

Let $\lambda \in (0,1)$ and

$$C_\lambda := \left\{ \sum_{j=0}^{\infty} a_j \lambda^j : a_j \in \{0,1,3\} \right\}.$$

C_λ is the self-similar set for the IFS

$$\{\varphi_1(x) = \lambda x,\, \varphi_2(x) = \lambda x + 1,\, \varphi_3(x) = \lambda x + 3\}.$$

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Two natural questions are

- What is the Hausdorff dimension of C_λ?
- Does C_λ have positive Lebesgue measure?

For $\lambda \in [2/5, 1)$ C_λ is an interval.

- What is the Hausdorff dimension of C_λ? - For Lebesgue almost every $\lambda \in (0, 1/3)$ we have $\dim_H C_\lambda = \frac{\log 3}{-\log \lambda}$. (Pollicott and Simon, 1995)

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Further results of Hochman, and Shmerkin and Solomyak yield that the set of exceptions to these statements has Hausdorff dimension zero.

As a by product of our methods we can give another proof of Soloymak's result.

We can show that for Lebesgue almost every $\lambda \in [1/3, 2/5)$ there exists $c_1, c_2 > 0$ such that for infinitely many $n \in \mathbb{N}$ there exists $S_n \subset \{0,1,3\}^n$ satisfying

- $\#S_n \geq c_1 \cdot 3^n$
- For $(a_j), (b_j) \in S_n$ we have

$$\left| \sum_{j=0}^{n-1} a_j \lambda^j - \sum_{j=0}^{n-1} b_j \lambda^j \right| > \frac{c_2}{3^n}.$$

For one of these λ we have the following:

$$\begin{aligned}\mathcal{L}(C_\lambda) &\geq \mathcal{L}\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\bigcup_{(a_j)\in\{0,1,3\}^n}\left(\sum_{j=0}^{n-1}a_j\lambda^j-\frac{c_2}{3^n},\sum_{j=0}^{n-1}a_j\lambda^j+\frac{c_2}{3^n}\right)\right)\\
&= \lim_{N\to\infty}\mathcal{L}\left(\bigcup_{n=N}^{\infty}\bigcup_{(a_j)\in\{0,1,3\}^n}\left(\sum_{j=0}^{n-1}a_j\lambda^j-\frac{c_2}{3^n},\sum_{j=0}^{n-1}a_j\lambda^j+\frac{c_2}{3^n}\right)\right)\\
&\geq \lim_{N\to\infty}\mathcal{L}\left(\bigcup_{n\geq N:S_n \text{ exists}}\bigcup_{(a_j)\in S_n}\left(\sum_{j=0}^{n-1}a_j\lambda^j-\frac{c_2}{3^n},\sum_{j=0}^{n-1}a_j\lambda^j+\frac{c_2}{3^n}\right)\right)\\
&\geq c_1\cdot 3^n\cdot\frac{2c_2}{3^n}=2c_1c_2>0\end{aligned}$$

A similar argument yields that almost surely the Bernoulli convolution is absolutely continuous.

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$$\left\{x \in \mathbb{R} : \left|x - \sum_{i=1}^{n} a_i \lambda^{i-1}\right| \leq \frac{1}{2^n} \text{ for i.m. } (a_1, \ldots, a_n) \in \cup_{m=1}^{\infty}\{-1, 1\}^m\right\}.$$

Thank you for listening.