Intermediate dimensions of Bedford–McMullen carpets

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Based on joint work with István Kolossváry [1]

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1. Hausdorff, box and intermediate dimensions

Hausdorff dimension is defined by covering a set with balls of varying sizes, whereas for box dimension all the sets in the cover must have equal size.

For $\theta \in (0,1)$, the *θ-intermediate dimension* of a non-empty, bounded set $F \subseteq \mathbb{R}^d$ is defined in [3] by

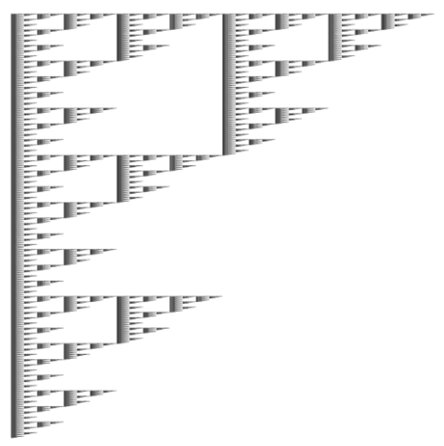
$$\overline{\dim}_\theta F = \inf\{\, s \geq 0 : \text{ for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

In particular $\dim_0 F = \dim_{\mathrm{H}} F$, $\overline{\dim}_1 F = \overline{\dim}_{\mathrm{B}} F$, and $\dim_{\mathrm{H}} F \leq \overline{\dim}_\theta F \leq \overline{\dim}_{\mathrm{B}} F$ for all $\theta \in (0,1)$. The function $\theta \mapsto \overline{\dim}_\theta F$ is increasing. It is continuous for $\theta \in (0,1]$. The intermediate dimensions are an example of dimension interpolation.

2. Bedford–McMullen carpets

Divide a square into an $m \times n$ grid, where $m < n$. Write $\gamma := \log n / \log m$. Let M be the number of non-empty columns, let N_i be the number of maps in the ith non-empty column, and let $N := N_1 + \ldots + N_M$. The carpet is the attractor of the iterated function system $\{S_1, \ldots, S_N\}$, where

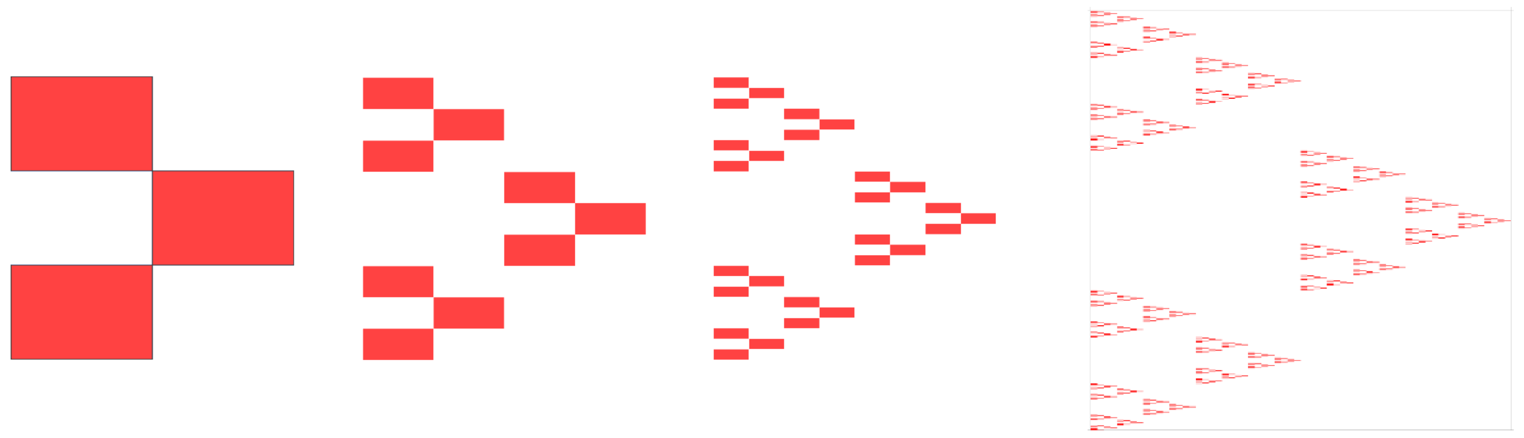
$$S_i(\underline{x}) := \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} (\underline{x}) + \underline{t}_i.$$



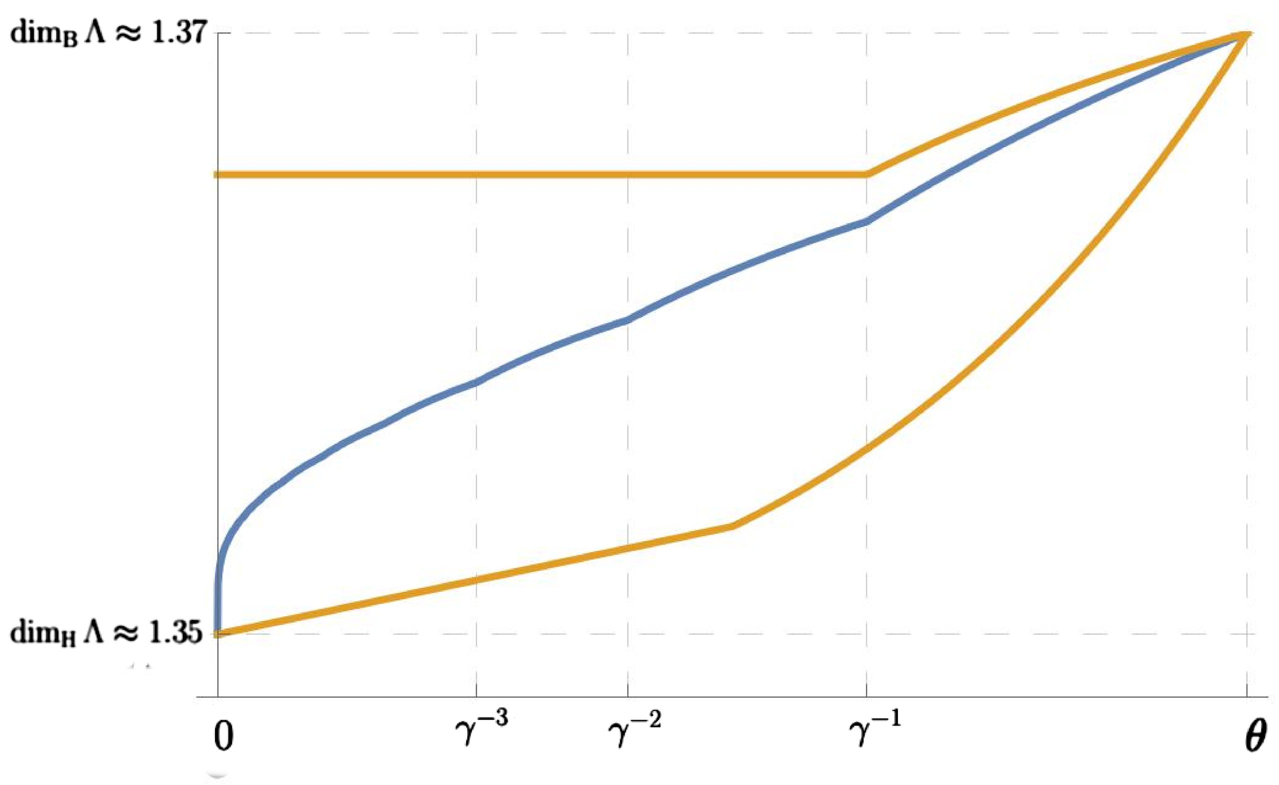
Bedford ('84) and McMullen ('84) independently proved that

$$\dim_{\mathrm{H}} \Lambda = \frac{1}{\log m} \log \left(\sum_{i=1}^{M} N_i^{\log m / \log n} \right); \qquad \dim_{\mathrm{B}} \Lambda = \frac{\log M}{\log m} + \frac{\log(N/M)}{\log n}.$$

In particular, $\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda$ if and only if Λ has uniform vertical fibres: $N_i = N/M$ for $i = 1, \ldots, M$. Throughout we assume this is not the case.



Cylinder sets for a Bedford–McMullen carpet



Some previous bounds for the intermediate dimensions (of the carpet shown to the left) in orange, and the true value in blue

3. Formula for the intermediate dimensions

Consider the large deviation rate function from probability theory defined by the following Legendre transform:

$$I(t) := \sup_{\lambda \in \mathbb{R}} \left\{ \lambda t - \log \left(\frac{1}{M} \sum_{j=1}^{M} N_j^{\lambda} \right) \right\}$$

For $s \in \mathbb{R}$, define the function $T_s \colon \mathbb{R} \to \mathbb{R}$ by

$$T_s(t) := \left(s - \frac{\log M}{\log m} \right) \log n + \gamma I(t).$$

For $\ell \in \mathbb{N}$, write $T_s^\ell := \underbrace{T_s \circ \cdots \circ T_s}_{\ell \text{ times}}$, and let T_s^0 be the identity map. Define

$$t_\ell(s) := T_s^{\ell-1} \left(\left(s - \frac{\log M}{\log m} \right) \log n \right).$$

Theorem (Main result of [1]). *Let Λ be any Bedford–McMullen carpet with non-uniform vertical fibres. For fixed $\theta \in (0,1)$ let $L = L(\theta) \in \mathbb{N}$ be such that $\gamma^{-L} < \theta \leq \gamma^{-(L-1)}$. Then there exists a unique solution $s = s(\theta) \in (\dim_{\mathrm{H}} \Lambda, \dim_{\mathrm{B}} \Lambda)$ to the equation*

$$\gamma^L \theta \log N - (\gamma^L \theta - 1) t_L(s) + \gamma (1 - \gamma^{L-1}\theta)(\log M - I(t_L(s))) - s \log n = 0,$$

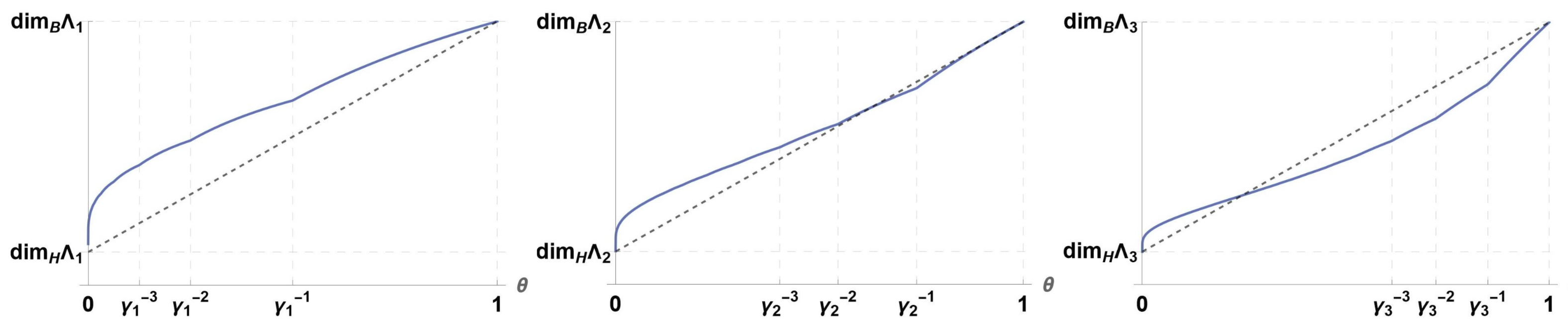
and $s(\theta) = \dim_\theta \Lambda$.

The proof of the upper bound involves constructing an explicit cover using the scales $\delta, \delta^\gamma, \delta^{\gamma^2}, \ldots, \delta^{\gamma^{L-1}}$ and $\delta^{1/\theta}, \delta^{1/(\gamma\theta)}, \ldots, \delta^{1/(\gamma^{L-1}\theta)}$, and we prove that for small θ we *need* to use more than two scales. The lower bound uses a mass distribution principle. The method of types is another key tool used.

4. Properties of the function $\theta \mapsto \dim_\theta \Lambda$

The function always has the following properties:

- Strictly increasing.
- Continuous for $\theta \in [0,1]$ (proved in [3]). Continuity at $\theta = 0$ has applications to box dimensions of orthogonal projections of carpets [2].
- Has phase transitions at integer powers of $1/\gamma$.
- Analytic and strictly concave between phase transitions.
- Slope tends to infinity as $\theta \to 0$.



Different possible shapes for the graph

5. Multifractal analysis and bi-Lipschitz equivalence

Let ν be the uniform Bernoulli measure supported on a Bedford–McMullen carpet. It gives mass $1/N$ to each first-level cylinder, and

$$\nu(A) = \sum_{i=1}^{N} \frac{1}{N} \nu(S_i^{-1} A) \quad \text{for all Borel sets } A \subset \mathbb{R}^2.$$

It is well-known that ν is exact dimensional: the local dimension

$$\dim_{\mathrm{loc}}(\nu, x) = \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}$$

exists and is constant at ν-almost every $x \in \Lambda$.

A formula for the multifractal spectrum of ν,

$$f_\nu(\alpha) := \dim_{\mathrm{H}} \{ x \in \operatorname{supp} \nu : \dim_{\mathrm{loc}}(\nu, x) = \alpha \},$$

is given in [4]. We combine this wth our main result to prove

Theorem. *If Λ, Λ' are Bedford–McMullen carpets with non-uniform vertical fibres, then the intermediate dimensions are equal for all θ if and only if the corresponding uniform Bernoulli measures have the same multifractal spectrum.*

If $f \colon \Lambda \to \Lambda'$ is bi-Lipschitz then it is straightforward to see that $\dim_\theta \Lambda = \dim_\theta \Lambda'$ for all θ. Therefore we can deduce the following:

Corollary. *If carpets Λ and Λ' with non-uniform vertical fibres are bi-Lipschitz equivalent then their uniform Bernoulli measures have the same multifractal spectra.*

This improves a result in [5]. We can also obtain bounds on the possible Hölder exponents of maps between carpets.

References

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