An heterogeneous mass transference principle, application to self-similar measure with overlaps

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Introduction

Mass transference principles or ubiquity theorems are tools designed to give lower-bounds for the Hausdorff dimension of sets which can be written as $\limsup_{n\to+\infty} U_n$, where U_n is typically a small closed ball or a small open set.

These sets arises naturally in Diophantine approximation and in dynamical systems and classical examples arises when one considers the rational numbers of $[0,1]$, $(x_n)_{n\in\mathbb{N}} = (\frac{p}{q})_{q\in\mathbb{N}^*, 0\leq p\leq q}$ in $\mathbb{R}$, in which case one studies the Hausdorff dimension $\limsup_{q\in\mathbb{N}^*,0\leq p\leq q} B(\frac{p}{q}, \frac{1}{q^\delta})$ or, given a measurable transformation $T: K\to K\subset\mathbb{R}^d$ and $x\in K$, one considers $(x_n)_{n\in\mathbb{N}} = (T^n(x))_{n\in\mathbb{N}}$, in which case one tries to compute $\dim_H(\limsup_{n\to+\infty} B(T^n(x), n^{-\delta}))$.

Historic of mass transference principles

A usual way to proceed to obtain lower-bounds of limsup sets is to find a measure $\mu\in\mathcal{M}(\mathbb{R}^d) := \{$ probability measure on $\mathbb{R}^d\}$ and a family of sets $(B_n)_{n\in\mathbb{N}}$ such that $x_n\in B_n$ and $\mu(\limsup_{n\to+\infty} B_n) = 1$. Then, one uses the geometric property of the measure μ to estimate $\dim_H(\limsup_{n\to+\infty} U_n)$ for sets $U_n\subset B_n$.

Theorem

Let us fix a sequence of balls $(B_n := B(x_n, r_n))_{n\in\mathbb{N}}$ satisfying $|B_n|\to 0$ and $\mu\in\mathcal{M}(\mathbb{R}^d)$.

1. **First result:** *if μ verifies $C_1 r^s \leq \mu(B(x,r)) \leq C_2 r^s$ and $\mu(\limsup_{n\to+\infty} B_n) = 1$, then Beresnevitch-Velani's theorem (see "A Mass Transference Principle and the Duffin-Schaeffer conjecture for Hausdorff measures") states that, for any $\delta > 1$ and any ball B with $\mu(B) > 0$,*
$$\mathcal{H}^{\frac{s}{\delta}}(B\cap \limsup_{n\to+\infty} B_n^\delta) = +\infty.$$
2. **Second result:**
Given any sequence of open sets $U_n\subset B_n$, if $\mathcal{L}^d(\limsup_{n\to+\infty} B_n) = 1$, then, Koivusalo and Rams (see "Mass transference principle: from ball to arbitrary shape") proved that for any s such that, for n large enough, $\mathcal{H}^s_\infty(U_n) := \inf\left\{\sum_{k\in\mathbb{N}} |A_k|^s, U_n\subset\bigcup_{k\geq 0} A_k\right\} \geq \mathcal{L}^d(B_n)$,
$$\dim_H(\limsup_{n\to+\infty} U_n) \geq s.$$
3. **Third result:** *When the measure μ is self-similar satisfying the open set condition, if $\mu(\limsup_{n\to+\infty} B_n) = 1$, then, by Barral-Seuret's theorem ("Heterogeneous ubiquitous systems in Rd and Hausdorff dimension")*
$$\dim_H(\limsup_{n\to+\infty} B_n^\delta) \geq \frac{\dim(\mu)}{\delta}.$$

μ-essential Hausdorff content

Definition

Let $\mu\in\mathcal{M}(\mathbb{R}^d)$. The s-dimensional μ-essential Hausdorff content of a set $A\subset\mathbb{R}^d$ is defined as
$$\mathcal{H}^{\mu,s}_\infty(A) = \inf\left\{\mathcal{H}^s_\infty(E), E\subset A \text{ and } \mu(E) = \mu(A)\right\}.$$

It turns out that the essential content is tractable for self-similar measures with no separation condition.

Theorem (D.)

Let μ be a self-similar measure. For any ball $B = B(x,r)$ centered on $K = supp(\mu)$ and $r\leq 1$, any open set Ω, one has
$$c(d,\mu,s)|B|^s \leq \mathcal{H}^{\mu,s}_\infty(\mathring{B}) \leq \mathcal{H}^{\mu,s}_\infty(B) \leq |B|^s \text{ and}$$
$$c(d,\mu,s)\mathcal{H}^s_\infty(\Omega\cap K) \leq \mathcal{H}^{\mu,s}_\infty(\Omega) \leq \mathcal{H}^s_\infty(\Omega\cap K).$$
For any $s > \dim(\mu)$, $\mathcal{H}^{\mu,s}_\infty(\Omega) = 0$.

An ubiquity theorem for probability measures

Theorem (mass transference principle for probability measures, D.)

Let $\mu\in\mathcal{M}(\mathbb{R}^d)$. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of balls satisfying $\mu(\limsup_{n\to+\infty}\frac{1}{2}B_n) = 1$ and let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open sets such that $U_n\subset B_n$.
Assume that, for $0\leq s\leq d$ and n large enough, $\mathcal{H}^{\mu,s}_\infty(U_n) \geq \mu(B_n)$, then
$$\dim_H(\limsup_{n\to+\infty} U_n) \geq s.$$

As a consequence, one gets the following result.

Corollary (mass transference principles for self-similar measures,D.)

Let S be a self-similar IFS of $\mathbb{R}^d$ with attractor K and μ be a self-similar measure associated with S. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of closed balls centered on K, such that $\lim_{n\to+\infty}|B_n| = 0$.
Suppose that $\mu(\limsup_{n\to+\infty} B_n) = 1$. Then, for any $\delta > 1$,
$$\dim_H(\limsup_{n\to+\infty} B_n^\delta) \geq \frac{\dim(\mu)}{\delta}.$$

Application to self-similar shrinking targets

Let $m\geq 2$ be an integer. Let $S = \{f_1, ..., f_m\}$ be a self-similar IFS of a compact X with contraction ratio $0 < c_1 \leq ... \leq c_m < 1$ and attractor K. Denote also $\Lambda = \{1, ..., m\}$ and $\Lambda^* = \bigcup_{k\geq 0}\Lambda^k$.
Define the similarity dimension $\dim(S)$ as the unique real satisfying
$$\sum_{i=1}^m c_i^{\dim(S)} = 1.$$

Theorem (dimension of self-similar shrinking targets,D.)

Assume $\dim_H(K) = \dim(S)$, then, for any $x\in K$ and any $\delta\geq 1$ it holds that
$$\dim_H\left(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x), |f_{\underline{i}}(K)|^\delta)\right) = \frac{\dim_H(K)}{\delta}, \tag{1}$$
where $f_{(i_1,...,i_k)} = f_{i_1}\circ ...\circ f_{i_k}$.

Theorem (complements to Baker's Theorem, D.)

Let $g:\mathbb{N}\to(0,+\infty)$ a non increasing mapping, define
$$s_g = \inf\left\{s\geq 0 : \sum_{k\geq 0}\sum_{\underline{i}\in\Lambda^k} k(|f_{\underline{i}}(K))|g(k))^s < +\infty\right\}.$$
Assume that $\dim(S) = \dim_H(K)$ and
$$\begin{cases} \sum_{i=1}^m -c_i^{\dim(S)}\log(c_i^{\dim(S)}) < -2\log\left(\sum_{i=1}^m c_i^{2\dim(S)}\right) \text{ or} \\ S \text{ is equicontractive.}\end{cases}$$
Then, for any $\delta\geq 1$,
$$\dim_H\left(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x), (|f_{\underline{i}}(K)|g(|\underline{i}|)^{\frac{\delta s_g}{\dim(S)}}))\right) = \frac{\dim_H(K)}{\delta}.$$

Some remark

- The estimates of the essential Hausdorff content in the self-similar case also allows to deal with other sets than shrunk balls. For instance it is possible to establish a mass transference principle from ball to rectangle for self-similar measures fully supported.
- Similar estimates of the essential Hausdorff content actually holds for measures associated with weakly conformal C^1 IFS's, so that the mass transference for self-similar measures also holds for those measures.