# Critical intermittency in random interval maps

Workshop on affine and overlapping IFSs

University of Bristol May 10-12, 2022

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# Good maps and bad maps

Homburg, Kalle, Ruziboev, Verbitskiy, Z. (2021):

We consider two types of maps, **good maps** and **bad maps**.

The class of good maps consists of maps

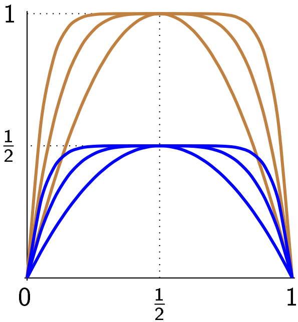
$$T_g : [0,1] \to [0,1], \quad T_g(x) = 1 - |1-2x|^{r_g},$$

where $r_g \geq 1$.

The class of **bad maps** consists of maps

$$T_b : [0,1] \to [0,1], \quad T_b(x) = \frac{1}{2} - \frac{1}{2}|1-2x|^{\ell_b},$$

where $\ell_b > 1$, called the *critical order*.



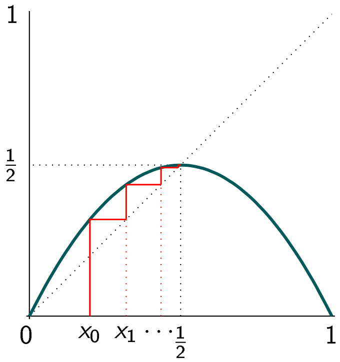

Ex: $r_g = 2, \ell_b = 2$ give $T_g(x) = 4x(1-x)$, $T_b(x) = 2x(1-x)$.

# Dynamics under bad maps

The orbits under a bad map $T_b$ are of the form $\{x_n\}_{n=0}^{\infty}$ where $x_0 \in [0,1]$ and, for each $n \in \mathbb{N}$,

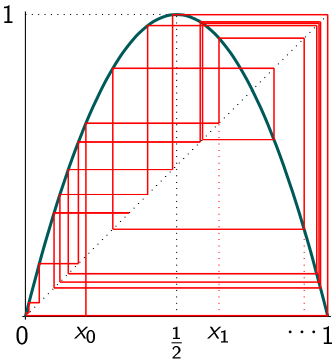
$$x_n = T_b(x_{n-1}) = \cdots = T_b^n(x_0) = \frac{1}{2} - \frac{1}{2}|1 - 2x_0|^{\ell_b^n}.$$

Hence, each bad map has $\frac{1}{2}$ as **superattracting fixed point** with $(0,1)$ as basin of attraction.



# Dynamics under good maps

The following result is a special case of the Main Theorem in Nowicki, van Strien (1991).




## Theorem

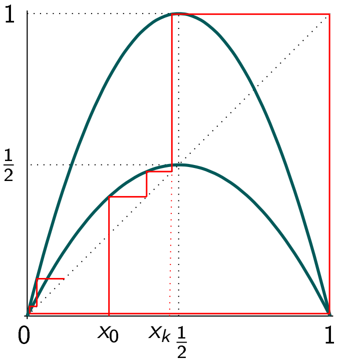
*Let $T_g$ be a good map.*

1. *There exists a unique $T_g$-invariant, $T_g$-ergodic probability measure $\mu_g$ such that $\mu_g \ll Leb$, where $Leb$ is the Lebesgue measure on $[0,1]$.*
2. *The density $\frac{d\mu_g}{dLeb}$ is bounded away from zero, is locally Lipschitz on $(0,1)$ and is in $L^q$ if and only if $q \in [1, \frac{r_g}{r_g-1})$.*

# Critical intermittency

Let $T_1, \ldots, T_N$ be a finite collection of good and bad maps, containing at least one of either type. Let $(p_j)_{j=1}^N$ be a prob. vector. Let $x_0 \in [0,1]$ and for each $n \in \mathbb{N}$, independently,

$$x_n = \begin{cases} T_1(x_{n-1}), & \text{with probability } p_1, \\ \vdots & \vdots \\ T_N(x_{n-1}), & \text{with probability } p_N, \end{cases}$$

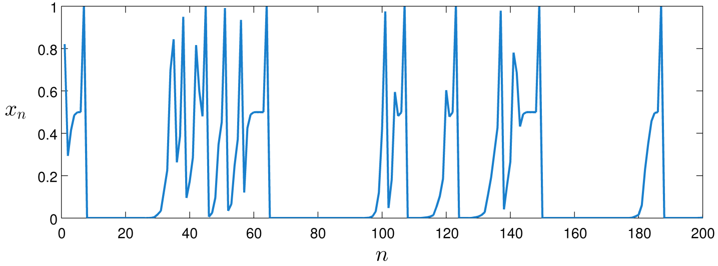



The typical behaviour is the following:

- The orbit **converges superexp. fast to $\frac{1}{2}$** under applications of bad maps up to some time $k$,
- At time $k+1$, the orbit is mapped close to 1 by a good map,
- Then the orbit is mapped close to 0 and **diverges exp. fast from 0** under applications of either good or bad maps.

# Critical intermittency

Time series of the random model consisting of the logistic maps $T_4(x) = 4x(1-x)$ and $T_2(x) = 2x(1-x)$, both chosen with equal probability at each timestep.




Abbasi, Gharaei, Homburg (2018): Random interval map composed of $T_4$ and $T_2$ admits a $\sigma$-finite acs measure $\mu$. Furthermore, $\mu$ is infinite in case $p_2 > \frac{1}{2}$, i.e. choosing $T_2$ with higher probability.

## Phase transition

Let $T_1, \ldots, T_N$ be a finite collection of good and bad maps, containing at least one of either type. Let $(p_j)_{j=1}^N$ be a prob. vector. A Borel measure $\mu$ on $[0,1]$ is *stationary* if

$$\sum_{j=1}^{N} p_j \mu(T_j^{-1} A) = \mu(A), \qquad \forall A \subseteq [0,1] \text{ Borel.}$$

**Theorem (Homburg, Kalle, Ruziboev, Verbitskiy, Z. (2021))**

1. *There exists a unique (up to scalar multiplication) $\sigma$-finite stationary measure $\mu$ such that $\mu \ll$ Leb.*
2. *The density $\frac{d\mu}{d\,Leb}$ is bounded away from zero, is locally Lipschitz on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and is not in $L^q$ for any $q > 1$.*
3. *Set $\Sigma_B = \{1 \leq j \leq N : T_j$ bad map$\}$ and $\theta = \sum_{b \in \Sigma_B} p_b \ell_b$. Then $\mu_{\mathrm{p}}$ is finite if and only if $\theta < 1$.*

Hence, the random system exhibits a **phase transition at** $\theta = 1$.